Adaptive Sampling Rule For Ranking-and-Selection Problem

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***Abstract*— If the variance are unequal among systems, it is reasonable to allocate samples unequally in ranking and selection procedures. In this paper, we propose two procedures allocating samples according to sample mean and variance information, namely the known variance adaptive sampling (KVAS) procedure and the unknown variance adaptive sampling (UVAS) procedure. Our procedure can reduce total sample size and still guarantee a pre-determined probability of correct selection. The experiment results show that (1)our proposed KVAS can reduce sample size nearly 20% compared with known variance unequal sample size procedure KVP provided in [12]; (2) our proposed UVAS can reduce total sample size between 30% and 50% compared with equal sample size procedure KN provided in [15] and unequal sample size procedure UVP provided in [12].**

I. INTRODUCTION

Ranking and selection (*R&S*) problem is to choose a system with largest or smallest mean value from a set of systems by simulation. There are two general kinds of approaches to solve *R&S* problem: a frequentist approach(see [2] for a summary) and a Bayesian approach. The key difference between these two approaches is that frequentist approach guarantees to select the best system with a pre-determined probability of correct selection and Bayesian approach devotes to maximize the posterior probability of correct selection given a finite computation budget by the art of allocation samples (see [6], [5], [4], [11],[3], [8], [7]as examples). Most frequentist approach procedures adopt the indifference-zone formulation proposed by [1]. If the difference between two systems is smaller than indifference-zone parameter δ, choosing any of these two systems are a good selection. Typical procedures are the two-stage procedure of [18] and the fully sequential procedure KN of [15]. Recently, [10] proposed a indifference-zone free procedure to relax this assumption. Our proposed procedures belong to frequentist approach and adopt the indifference-zone parameter.

Another difference between the two approaches concerns the sample size allocation. For Bayesian approach, procedures always allocate different number of observations to systems to maximize probability of correct selection. However, for frequentist approach, procedures always use vector-filling method, for example [18]. [16] procedure and KN procedure. Early works in this stream of research consider equal variance or known variance cases (see for example [17] and [9]) and the focus is to improve the effectiveness of procedures. Thus, compared with Bayesian approach, frequentist procedure requires more observations because systems with larger variance need more samples to estimate sample mean. [19], [13] are among the early works that consider unequal sample size procedure. For unequal sample size procedure, the first question is how to allocate observations to different systems. In order to obtain the statistical validity, the sampling rule can only depends on the differences between systems. Otherwise, we no longer have the Brownian motion approximation property, which is critical for the proof of statistical validity. However, [14] showed that if sampling rule depended on sample mean information, it still performed well in the view of probability of correct selection. [12] considered Known-variance procedure and Unknown-variance procedure that allocate different number of observations to different system based on the relationship of variance. He show that the sampling rule should satisfy

$$\frac{n_1}{\sigma_1} = \frac{n_2}{\sigma_2} = \cdots = \frac{n_k}{\sigma_k}.$$

The procedure he proposed is the first one that can not only guarantee finite-time statistically valid but also implement unequal sample size. He proved that unequal sample size procedure reduces total sample significantly compared with KN by simulation. However, for the case of unknown variance case, his method adopted the triangular boundary proposed by [17] which is inferior to the boundary of [9]. Although according to his numerical study, using [9]'s boundary still can select the best system with satisfactory probability, the statistical validity can not be obtained theoretically.

Through these retrospect, we found that: 1. unequal sample size procedure is better than equal sample size procedure; 2. generally, using unequal sample size method can still insure the validity; 3. if the sampling rule depends on mean information, even though it is hard to have theoretical proof of statistical validity, it still works well in numerical. Motivated by these findings, we present a new adaptive sampling rule which depends not only on variance but also on mean information. For known variance scenario, we can prove the statistical validity in finite-time. For unknown variance, we prove that we can still choose the best system with required probability asymptotically if the indifference-zone parameter δ goes to 0 and sample size goes to infinity. Sampling rule in [13] and [12] could be regarded as a special case of our method in some sense. We discuss this in section 2.1.

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Our contribution are: 1. the first procedure of which sampling rule depends on both mean and variance information; 2. our unknown variance procedure can update sample variance continuously; 3. we prove the statistical validity in finite-time(known variance case) and asymptotically(unknown variance case).

In this paper, we present our procedure and sampling rule in Section 2. To improve performance, we present two procedures for both known and unknown variance cases. When discuss sampling rule, we use slippage configuration case as an illustration. Then we extend it to general configuration. In Section 3, we prove that our construction can be approximate by a standard Brownian motion process with a drift. In Section 4, we prove the statistical validity. A comprehensive numerical study is presented in Section 5 to check the validity and effectiveness. In Section 6, we conclude our paper.

II. PROCEDURE

Our aim is to choose the system with the largest mean value from several systems. Without loss of generality, we assume that there are k systems and $\mu_1 - \delta \geq \mu_2 \geq \cdots \geq \mu_k$, where δ is the indifference-zone parameter. Each system is normally distributed with mean μ_i and variance σ_i^2. Any two systems are bivariate normal. Denote the lth sample of system i as X_{il}. X_{il}, for $l = 1,2,\ldots,$ are independent and identically distributed. Sample mean and variance are $\bar{X}_i$ and S_i^2 for system i.

In procedure KVAS, we assume systems variance are known before running the procedure but they could be different among systems. Compared with other procedure like KN, the most significant modification is that this is not a vector-filling procedure any more, which means sample size among different systems could be unequal. Briefly, the relationship between sample size of each system relies on their sample variance, which is described in detail in section II-A. This allows the procedure to be more efficient by utilizing the information of sample variance. What is more, this procedure is different with other unequal sample size procedure such as UVP in [12] in the view of sampling rule. UVP consider the sample size proportion relationship between any two systems. However, we solve an optimization problem which takes all the systems into consideration.

PROCEDURE 1 (KNOWN VARIANCE ADAPTIVE SAMPLING (KVAS) PROCEDURE)

Step 1. *Setup.* Select overall confidence level $0 < 1-\alpha < 1$ and indifference zone parameter δ. Let $I = \{1,2,...,k\}$ be the original set of systems. Obtain one observation for each system i. Set counter $r = k$. Calculate a by solving the following equation:

$$a_1 = -\frac{1}{\delta}\log[2-2(1-\alpha)^{\frac{1}{k-1}}]$$

Step 2. *Comparison.* For all the system i still in contending, allocate the next observation by the sampling rule described in section II-A. Update mean value and let $r = r+1$. Set $I_{old} = I$. Let

$$I = I_{old} \setminus \{i : i \in I_{old} \text{ and } Z_{li}(t_{li}) \geq \max\{0, a_1 - \frac{\delta}{2}t_{li}\} \quad \forall l \in I^{old}, l \neq i\}$$

Step 3. *Stopping Rule.* If $|I = 1|$, stop and select the system whose index is in I as the best. Otherwise, go back to *Comparison*.

In procedure UVAS, we relax the restriction about variance. It is unknown to the experimenter and also could be unequal. Because the variance is unknown, we change the construction of the means difference sequence and the triangular region boundary in order to insure the statistical validity. To estimate sample variance, we need an initial stage in which we take n_0 samples for each system. Another notable modification is that sample variance could be updated during the procedure. As it is known to all, updating sample variance can improve not only the effectiveness but also the accuracy.

PROCEDURE 2 (UNKNOWN VARIANCE ADAPTIVE SAMPLING (UVAS) PROCEDURE)

Step 1. *Setup* Select $n_0 \geq 2$, overall confidence level $0 < 1-\alpha < 1$ and indifference zone parameter δ. Let $I = \{1,2,...,k\}$ be the original set of systems. Obtain n_0 observations X_{ij}, $j = 1,2,...,n_0$, for each system i. Calculate sample variance S_i^2 and sample mean $\bar{X}_i(n_0)$ for each system. Calculate a by solving the following equation:

$$a_2 = -\log[2-2(1-\alpha)^{\frac{1}{k-1}}]$$

Step 2. *Comparison.* For all the systems still in contending, allocate the next observation by the sampling rule described in section II-A. If the observation is allocated to system i, update mean value and variance of each system and let $n_i = n_i + 1$. Set $I_{old} = I$. Let

$$I = I_{old} \setminus \{i : i \in I_{old} \text{ and } Y_{li}(\tau_{li}) \geq \max\{0, a_2/\delta - \delta\tau_{li}/2\} \quad \forall l \in I^{old}, l \neq i\}.$$

Step 3. *Stopping Rule.* If $|I = 1|$, stop and select the system whose index is in I as the best. Otherwise, go back to *Comparison*.

A. Sampling Rule

1) Slippage Configuration: In this section, we present the sampling rule with slippage configuration and unequal variance case and its derivation process. In summary, we work out the optimal relationship between system by minimizing total sample size needed by the procedure. Denote n_i as the sample size needed by system i, for $i = 1,2,\ldots,k$. Without loss of generality, we assume that $\mu_1 - \delta = \mu_2 = \cdots = \mu_{k-1} = \mu_k$. The first exit time for $\mathbf{B}_{\mu_1-\mu_i}(t)$ to leave the triangular region is t_{1i}. To minimize the total sample size, we have the optimization function below.

$$\begin{aligned} &\min n_1+n_2+\cdots+n_k, \\ &s.t.\ \left[\frac{\sigma_1^2}{n_1}+\frac{\sigma_i^2}{n_i}\right]^{-1} = t_{1i},\ \text{for } i=2,3,\ldots,k. \end{aligned} \tag{II.1}$$

By the definition of slippage configuration, $\tau_{1i}=t_{1j}$, for $i,j=2,3,\ldots,k$ and $i\neq j$, we obtain that $n_j=\sigma_j^2 n_i/\sigma_i^2$. So the primal problem can be transfered to

$$\begin{aligned} \min\quad & \frac{\sum_{i=2}^k \sigma_i^2}{\sigma_1^2} n_2 + n_1, \\ s.t.\quad & \left[\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}\right]^{-1} = t_{12}. \end{aligned}$$

Denote λ as the Lagrange Multiplier. The Lagrange function is

$$L(n_1,n_2,\lambda)=\frac{\sum_{i=2}^k \sigma_i^2}{\sigma_2^2} n_2 + n_1 + \lambda\left(\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}-\frac{1}{\tau_{12}}\right).$$

By the first order condition, we have

$$\begin{aligned} \frac{\partial L}{\partial n_1} &= \frac{\sum_{i=2}^k \sigma_i^2}{\sigma_2^2} - \frac{\lambda\sigma_1^2}{n_1^2} = 0, \\ \frac{\partial L}{\partial n_2} &= 1 - \frac{\lambda\sigma_2^2}{n_2^2} = 0. \end{aligned}$$

Then we obtain the relationship between the sample size of each system

$$\begin{aligned} &\frac{n_i}{\sigma_i^2} = \frac{n_j}{\sigma_j^2},\ \text{for } i,j=2,3,...,k \text{ and } i\neq j \\ &\frac{n_1}{\sigma_1} = \sqrt{\sum_{i=2}^k \frac{n_i^2}{\sigma_i^2}}. \end{aligned} \tag{II.2}$$

To minimize total sample size, during the procedure, we want to maintain the relationship between system as described above. So the sampling rule is to allocate the next sample to the system with smallest n_i/σ_i^2.

Remark 1: If $\sigma_1^2=\sigma_2^2=\cdots=\sigma_k^2$, we have $n_1:n_2:\cdots:n_k=\sqrt{k-1}:1:\cdots:1$. This is consistent with the result in [13].

2) General Configuration: In this section, we consider general configuration. We do not have any restrictions on the relationship on mean value. Slippage configuration is a special case of this. Denote n_i as the sample size needed by system i, for $i=1,2,\ldots,k$. Without loss of generality, we assume that mean value for system i is μ_i and system 1 is the best system. The expected first exit time for $\mathbf{B}_{\mu_1-\mu_i}(t)$ to leave the triangular region is

$$\tau_{1i}=\frac{a}{\mu_1-\mu_i+\lambda}$$

.

To minimize the total sample size, we have the optimization function below.

$$\begin{aligned} \min\quad & n_1+n_2+\cdots+n_k, \\ s.t.\quad & \left[\frac{\sigma_i^2}{n_i}+\frac{\sigma_1^2}{n_1}\right]^{-1} = \tau_{1i},\ \text{for } i=2,3,\ldots,k. \end{aligned}$$

For any two system i and j, if $i\neq j\neq 1$, we have

$$\frac{\sigma_i^2}{n_i}-\frac{\sigma_j^2}{n_j}=\frac{\mu_j-\mu_i}{a}.$$

This relationship here is more complicated than the one in slippage configuration because we not only need to consider the influence of variance but also pay close attention to mean value. For unknown variance case, the sampling rule could be summarized as

1) Sort the systems by sample mean. Denote the system with largest mean as system b and the system with second larest mean as system s.
2) In order to insure optimal sample size for each system is a positive number, calculate the optimal sample size proportion using system s as a basis.

$$n_i' = \begin{cases} \left\lceil \frac{\sigma_i^2}{\sigma_s^2/n_s+(\bar{X}_s-\bar{X}_i)/a} \right\rceil, & \text{for } i\neq s \text{ and } i\neq b \\ \left\lceil \sqrt{\sum_{j=1,j\neq b}^k \frac{n_j^2\sigma_b^2}{\sigma_j^2}} \right\rceil, & \text{for } i=b \end{cases}$$

3) Allocate the next sample to system i still in contention with smallest n_i/n_i'. If $n_i/n_i'=n_j/n_j'$, allocate the next observation to the system with smaller variance. If $n_i/n_i'=1$ for all system $i\in I$, take one sample for the system with smallest variance.

Remark 2: If variance are unknown, substitute σ_i^2 by S_i^2. Also, in our procedure, variance can be updated after taking one more observation, so in order to exploit the full potential, we recommend calculate the sampling rule by updating variance. If there are only two systems, the relationship should be $n_i/\sigma_i=n_j/\sigma_j$. This is equivalent to the result in [12] because [12] considered a minimization problem of total sample size for any two systems.

III. Construction of Brownian Motion Process

In this paper, we consider two different approaches to construct the approximation of the Brownian motion process for known and unknown variance scenarios. In Lemma 1, we proved that for known variance case, the sequences $Z_{ij}(t_{ij})$ has the same distribution as a standard Brownian motion process with drift $\mu_i-\mu_j$. When talking about unknown variance, it is hard to construct the Brownian motion process in finite time perspective. In Lemma 2, we proved that sequence $Y_{ij}(\tau_{ij})$ has the same distribution as a standard Brownian motion process with drift $\Delta=\sqrt{2a_2}(\mu_i-\mu_j)/\delta$ asymptotically. Denote $\mathbf{B}_\delta(t)$ as a Brownian motion process with drift δ.

First, if variance is known, for any two systems i and j, we construct the sequences

$$\begin{aligned} Z_{ij}(t_{ij}) &= t_{ij}[\bar{X}_i(n_i)-\bar{X}_j(n_j)] \\ t_{ij} &= \left[\frac{\sigma_i^2}{n_i}+\frac{\sigma_j^2}{n_j}\right]^{-1} \end{aligned}$$

where $\bar{X}_i(n_i)=\frac{\sum_{l=1}^{n_i} X_{il}}{n_i}$. Although this is an unequal sample size procedure, if variance is equal, it is equivalent to the equal sample size procedure. So equal sample size procedure

like KN could be regarded as a special case of this. In the following Lemma, we prove $Z_{ij}(t_{ij})$ has the same distribution as $\mathbf{B}_{\mu_i-\mu_j}(t_{ij})$. This proof is similar as Theorem 1 in [12].

Lemma 1: For any two system i and j, the random non-decreasing sequence $Z_{ij}(t_{ij})$ has the same distribution with $\mathbf{B}_{\mu_i-\mu_i}(t_{ij})$.

Proof: In order to prove the random non-decreasing sequence $Z_{ij}(t_{ij})$ has the same distribution with $\mathbf{B}_{\mu_i-\mu_i}(t_{ij})$, recall that

1) $\mathbf{B}_{\mu_i-\mu_i}(0)=0$;
2) For any $0<t_p<\cdots<t_q<\infty$, $\mathbf{B}_{\mu_i-\mu_i}(t_p),\ldots,\mathbf{B}_{\mu_i-\mu_i}(t_q)$ are jointly normally distributed;
3) For any $0<t_{ij}<\infty$, $\mathrm{E}[\mathbf{B}_{\mu_i-\mu_i}(t_{ij})]=t_{ij}(\mu_i-\mu_i)$, and $\mathrm{Var}[\mathbf{B}_{\mu_i-\mu_i}(t_{ij})]=t_{ij}$.
4) for $0<u_{ij}<v_{ij}<+\infty$, $\mathrm{Cov}(\mathbf{B}_{\mu_i-\mu_i}(u_{ij}),\mathbf{B}_{\mu_i-\mu_i}c(v_{ij}))=u_{ij}$.

By the definition of $Z_{ij}(t_{ij})$, we have $Z_{ij}(0)=0$. Since $Z_{ij}(t_{ij})$ are linear function of jointly normal random variables, they are jointly normally distributed. Then

$$\mathrm{E}[Z_{ij}(t_{ij})]=t_{ij}(\mu_i-\mu_j),\mathrm{Var}[Z_{ij}(t_{ij})]=t_{ij}.$$

With these two properties, it suffices to prove that $\mathrm{Cov}(\mathbf{B}_{\mu_i-\mu_i}(u_{ij}),\mathbf{B}_{\mu_i-\mu_i}c(v_{ij}))=u_{ij}$, for $0<u_{ij}<v_{ij}<+\infty$. Without loss of generality, we assume system i and j obtain m and n samples at time u_{ij}, respectively. System i and j obtain p and q samples between u_{ij} and v_{ij}, respectively.

Then we can rewrite $Z_{ij}(u_{ij})$ and $Z_{ij}(v_{ij})$ as

$$\begin{aligned}Z_{ij}(u_{ij}) &= \left(\frac{\sigma_i^2}{m}+\frac{\sigma_j^2}{n}\right)^{-1}(\bar{X}_{i(u)}-\bar{X}_{j(u)})\\ Z_{ij}(v_{ij}) &= Z_{ij}(u_{ij})+\left(\frac{\sigma_i^2}{p}+\frac{\sigma_j^2}{q}\right)^{-1}(\bar{X}_{i(v)}-\bar{X}_{j(v)})\end{aligned}$$

Because sequence Z_{ij} is $i.i.d$, and a and b are independent. Since $\mathrm{Cov}\left(Z_{ij}(u_{ij}),\left(\frac{\sigma_i^2}{p}+\frac{\sigma_j^2}{q}\right)^{-1}(\bar{X}_{is(v)}-\bar{X}_{js(v)})\right)=0$, we have $\mathrm{Cov}(Z_{ij}(u_{ij}),Z_{ij}(v_{ij}))=u_{ij}$ ■

If variance is unknown, we construct the sequences as

$$\begin{aligned}Y_{ij}(\tau_{ij}) &= \tau_{ij}[\bar{X}_i(n_i)-\bar{X}_j(n_j)]\\ \tau_{ij} &= \left[\frac{S_i^2}{n_i}+\frac{S_j^2}{n_j}\right]^{-1}\end{aligned}$$

where

$$\begin{aligned}\bar{X}_i(n_i) &= \frac{\sum_{l=1}^{n_i}X_{il}}{n_i}\\ S_i^2 &= \frac{1}{n_0-1}\sum_{l=1}^{n_0}[X_{ij}-\bar{X}_i(n_0)]^2\end{aligned}$$

for $i,j=1,2,...,k$ and $i\neq j$. In this formulation, we no longer have the properties of variance and covariance. We need construct a new sequence as described below.

Lemma 2: Under asymptotic condition $\delta\to 0$, $n_0(\delta)\to\infty$, $\delta^2 n_0\to 0$, and $\tilde{Y}_{ij}(s)$ has the same distribution as $\mathbf{B}_\Delta(s)$ asymptotically, where $\Delta=\sqrt{2a_2}(\mu_i-\mu_j)/\delta$.

Proof: Denote N_{ij} as the maximum sample size needed for the comparison of system i and j for the known variance scenario. Then for system i and j, we can rewrite the sample size obtained as $n_i=\lfloor Nx_is\rfloor$ and $n_j=\lfloor Nx_js\rfloor$, respectively. Let x_i be the proportion of system i and let s be a parameter in $[0,1]$. In this procedure, the triangular region boundary is formed by $(-a_2/\delta+\delta t_{ij}/2, a_2/\delta-\delta t_{ij}/2)$, we have the maximum time $t_{ij}=2a_2/\delta^2$. Notice that $t_{ij}=(\sigma_i^2/n_i+\sigma_j^2/n_j)^{-1}$, then the maximum sample size needed by system i and j could be calculated by $N_{ij}=\frac{2a_2(\sigma_i^2/x_i+\sigma_j^2/x_j)}{\delta^2}$ Define

$$\tilde{Y}_{ij}(s) = \left[s\sqrt{\frac{N_{ij}}{\sigma_i^2/x_i+\sigma_j^2/x_j}}(\sigma_i^2/n_i+\sigma_j^2/n_j)\right]Y_{ij}(\tau_{ij})$$

Because $\tilde{Y}_{ij}(s)$ is a linear function of jointly normal random variables, they are also jointly normally distributed. Obviously, $\tilde{Y}_{ij}(0)=0$. The asymptotic regime is $\delta\to 0$, $n_0(\delta)\to\infty$. Then by the law of large numbers, $\frac{\sigma_i^2/n_i+\sigma_j^2/n_j}{S_i^2/n_i+S_j^2/n_j}\to 1$ with probability 1. Besides, we can rewrite the means difference as

$$\begin{aligned}[\bar{X}_i(n_i)-\bar{X}_j(n_j)] &= \frac{1}{n_i}\sum_{l=1}^{n_i}X_{il}-\frac{1}{n_j}\sum_{l=1}^{n_j}X_{jl}\\ &= \frac{1}{Nx_is}\sum_{l=1}^{Nx_is}X_{il}-\frac{1}{Nx_js}\sum_{l=1}^{Nx_js}X_{jl}\end{aligned}$$

So we have

$$\begin{aligned}\tilde{Y}_{ij}(s) &= s\sqrt{\frac{N}{\sigma_i^2/x_i+\sigma_j^2/x_j}}\left(\frac{1}{Nx_is}\sum_{l=1}^{Nx_is}X_{il}-\frac{1}{Nx_js}\sum_{l=1}^{Nx_js}X_{jl}\right)\\ &= \sqrt{\frac{N_{ij}}{\sigma_i^2/x_i+\sigma_j^2/x_j}}\left[\frac{\sigma_i}{\sqrt{Nx_i}}\frac{1}{\sigma_i\sqrt{Nx_i}}\sum_{l=1}^{Nx_is}(X_{il}-\mu_i)\right.\\ &\quad\left.-\frac{\sigma_j}{\sqrt{Nx_j}}\frac{1}{\sigma_j\sqrt{Nx_j}}\sum_{l=1}^{Nx_js}(X_{jl}-\mu_j)+s(\mu_i-\mu_j)\right]\\ &= \sqrt{\frac{N_{ij}}{\sigma_i^2/x_i+\sigma_j^2/x_j}}\\ &\quad\left[\frac{\sigma_i}{\sqrt{Nx_i}}\mathbf{B}^i(s)+\frac{\sigma_j}{\sqrt{Nx_j}}\mathbf{B}^j(s)+s(\mu_i-\mu_j)\right]\\ &\overset{\mathbb{D}}{=} \mathbf{B}(s)+\sqrt{\frac{N_{ij}}{\sigma_i^2/x_i+\sigma_j^2/x_j}}s(\mu_i-\mu_j)\\ &\quad\text{(since } \mathbf{B}^i(s) \text{ and } \mathbf{B}^j(s) \text{ are independent)}\\ &= \mathbf{B}(s)+\Delta s\end{aligned}$$

where $\Delta=\sqrt{2a_2}(\mu_i-\mu_j)/\delta$.

The forth equation is because by Donsker's Theorem ([21], Theroem 4.3.2), as $\delta\to 0$, $\frac{1}{\sigma_i\sqrt{Nx_i}}\sum_{l=1}^{Nx_is}(X_{il}-\mu_i)$ has the same distribution as a standard Brownian motion process.

So $\tilde{Y}_{ij}(s)$ has the same distribution of a standard Brownian motion process with drift $\Delta=\sqrt{2a_2}(\mu_i-\mu_j)/\delta$. This nice property can help us to design an unequal sample size procedure for unknown variance case. ■

IV. Statistical Validity

In this section, we prove the statistical validity of KVAS and UVAS procedure, that is the procedure can choose the best system with required probability of correct selection. For KVAS procedure, because of the nice property, we have the statistical validity in finite time. However, if variance are unknown, it is difficult to construct Brownian motion process approximation. The proof of KVAS achieves asymptotically statistical validity. At first, we need the following Lemma 3 and 4.

Lemma 3: ([9]) Let $\mathbf{B}_\delta(t)$ be a Brownian motion with drift $\delta > 0$. A triangular region Π is formed by $[-g(t), g(t)]$, where $g(t) = a - \lambda t$. Let $\tau = \inf\{t : \mathbf{B}_\Delta(t) \notin \Pi\}$, and ε be the event $\{\mathbf{B}_\delta(\tau) < -g(\tau)\}$. If $\lambda = \delta/2$, then

$$\Pr\{\varepsilon\} = \frac{1}{2}e^{-a\delta}$$

Lemma 4: ([13]) Consider $\mathbf{B}_\delta(t)$ on $[0, +\infty)$. Denote T_d as the first time that $\mathbf{B}_\delta(t)$ falls outside the triangular region $(-g(t), g(t))$ and T_c as the first time that $\mathbf{B}_\delta(t)$ falls outside the triangular region observed in discrete time interval $\{t_1, t_2, ..., \}$. Assume $T_d < \infty$ almost surely and t_i has the same conditional distribution, given $\mathbf{B}_\delta(t) = b$ and $\mathbf{B}_\delta(t) = -b$, then

$$\Pr\{\mathbf{B}_\delta(T_d) < -g(t)\} \le \Pr\{\mathbf{B}_\delta(T_c) < -g(t)\}.$$

Recall that we have already proved that the sequence $Z_{ij}(t_{ij})$ has the same distribution as $\mathbf{B}_{\mu_i - \mu_j}(t_{ij})$. With these above two lemma, we have the Theorem 1.

Theorem 1: Suppose that $X_{il}, l = 1, 2, \ldots,$ are i.i.d. normally distributed and X_{il} and X_{jl} are joint normal distributed, for any $i, j \in I$ and $i \neq j$. If variance are known, procedure KVAS can choose the best system with probability at least $1 - \alpha$.

Proof: Denote $T_{1l}^{1\delta}$, $T_{12}^{1\delta}$, and $T_{13}^{1\delta}$, as the first time that sequence $Z_{1i}(t_{ij})$, $\mathbf{B}_{\mu_1 - \mu_i}(t_{ij})$ and $\mathbf{B}_\delta(t_{ij})$ leave the continuation region. A wrong selection here means that the best system is eliminated by inferior systems. At first, we consider the probability of a wrong selection.

$$\begin{aligned}
& \Pr\{\text{system } i \text{ eliminates system } 1\} \\
= \; & \Pr\{Z_{1i}(T_{12}^{1\delta}) < 0\} \\
\le \; & \Pr\{\mathbf{B}_{\mu_1 - \mu_i}(T_{12}^{2\delta}) < 0\} \quad \text{by Lemma 4} \\
\le \; & \Pr\{\mathbf{B}_\delta(T_{12}^{3\delta}) < 0\} \quad \text{since } \mu_k - \mu_i \ge \delta \\
\le \; & \frac{1}{2}e^{-a_1\delta} \quad \text{by Lemma 3} \\
= \; & 1 - (1-\alpha)^{\frac{1}{k-1}} \text{ by the definition of } a_1
\end{aligned}$$

Assuming there are k systems, a correct selection means that system 1 is not eliminated by any inferior systems. By Bonferroni inequality, we have

$$\begin{aligned}
& \Pr\{\text{choose system 1 as the best system}\} \\
= \; & \prod_{i=2}^{k}(1 - \Pr\{\text{system 1 is eliminated by system } i\}) \\
= \; & \prod_{i=2}^{k}[1 - (1 - (1-\alpha)^{\frac{1}{k-1}})] \\
= \; & 1 - \alpha.
\end{aligned}$$

■

Now, we prove the asymptotically statistical validity of UVAS procedure. The difficulty is that the sequence $Y_{ij}(\tau_{ij})$ does not have the same distribution as a Brownian motion process. We need to define a new sequence $\tilde{Y}_{ij}(s)$ as mentioned before and this sequence has the same distribution as $\mathbf{B}_\Delta(s)$ asymptotically, where $\Delta = \sqrt{2a_2}\frac{\mu_i - \mu_j}{\delta}$.

Theorem 2: Suppose that $X_{il}, l = 1, 2, \ldots,$ are i.i.d. normally distributed and X_{il} and X_{jl} are joint normal distributed, for any $i, j \in I$ and $i \neq j$. Variance of each system are unknown. Under asymptotic condition, $\delta \to 0$, procedure UVAS can choose the best system with probability at least $1 - \alpha$.

Proof: The relationship between $Y_{ij}(\tau_{ij})$ and $\tilde{Y}_{ij}(s)$ is

$$\tilde{Y}_{ij}(s) = Y_{ij}(\tau_{ij})\left[s\sqrt{\frac{N_{ij}}{\sigma_i^2/x_i + \sigma_j^2/x_j}}(\sigma_i^2/n_i + \sigma_j^2/n_j)\right].$$

With the same multiplier, the triangular region boundary can be rewrite as

$$\begin{aligned}
& \left(\frac{a_2}{\delta} - \frac{\delta}{2}\tau_{ij}\right)\left[s\sqrt{\frac{N_{ij}}{\sigma_i^2/x_i + \sigma_j^2/x_j}}(\sigma_i^2/n_i + \sigma_j^2/n_j)\right] \\
= \; & \frac{a_2}{\gamma} - \frac{\gamma}{2}\frac{\sigma_i^2/n_i + \sigma_j^2/n_j}{S_i^2/n_i + S_j^2/n_j}s
\end{aligned}$$

where $\gamma = \sqrt{2a_2}$.

With the asymptotic condition $\delta \to 0$, $\frac{\sigma_i^2/n_i + \sigma_i^2/n_j}{S_i^2/n_i + S_i^2/n_j} \to 1$. So comparing $Y_{ij}(\tau_{ij})$ to $(-a_2/\delta + \delta\tau_{ij}/2, a_2/\delta - \delta\tau_{ij}/2)$ is equivalent to compare $\tilde{Y}_{ij}(s)$ to $(-a_2/\gamma + \gamma s/2, a_2/\gamma - \gamma s/2)$.

Denote $T_{ij}^{1\gamma}$ as the stopping time at which $Y_{ij}(\tau_{ij})$ exits the continuation region $(-a_2/\delta + \delta\tau_{ij}/2, a_2/\delta - \delta\tau_{ij}/2)$. Instead of analyzing $Y_{ij}(\tau_{ij})$, now consider sequence $\tilde{Y}_{ij}(s)$. Then an incorrect selection means that $\tilde{Y}_{1j}(s)$ leaves the new triangular region downward. Denote $T_{ij}^{2\gamma}$, $T_{ij}^{3\gamma}$ and $T_{ij}^{4\gamma}$ as the stopping time at which $\tilde{Y}_{ij}(s)$, $\mathbf{B}_\Delta(s)$, and $\mathbf{B}_\gamma(s)$ exits the continuation region $(-a_2/\gamma + \gamma s/2, a_2/\gamma - \gamma s/2)$,

respectively. Condition on S_1^2 and S_i^2, we have

$$\begin{aligned}
&\Pr\{\text{system } i \text{ eliminates system } 1\} \\
=\ & \mathrm{E}[\Pr\{\text{system } i \text{ eliminates system } 1|S_1^2,S_i^2\}] \\
=\ & \mathrm{E}[\Pr\{Y_{1j}(T_{1j}^{1\gamma})<0\}|S_1^2,S_i^2] \\
=\ & \mathrm{E}[\Pr\{\tilde{Y}_{1j}(T_{1j}^{2\gamma})<0\}|S_1^2,S_i^2] \\
\le\ & \mathrm{E}[\Pr\{\mathbf{B}_\Delta(T_{1j}^{3\gamma})<0\}|S_1^2,S_i^2] \text{ by Lemma 2 and 4} \\
\le\ & \mathrm{E}[\Pr\{\mathbf{B}_\gamma(T_{1j}^{4\gamma})<0\}|S_1^2,S_i^2] \text{ since } \mu_1-\mu_i\ge\delta \\
\le\ & \frac{1}{2}e^{(-a_2/\gamma)\gamma} \quad \text{by Lemma 3} \\
=\ & 1-(1-\alpha)^{\frac{1}{k-1}} \text{ by the definition of } a_2
\end{aligned}$$

Using the same technique as before, by Bonferroni inequality, we have

$$\begin{aligned}
&\Pr\{\text{choose system 1 as the best system}\} \\
=\ & \prod_{i=2}^{k}(1-\Pr\{\text{system 1 is eliminated by system } i\}) \\
=\ & \prod_{i=2}^{k}[1-(1-(1-\alpha)^{\frac{1}{k-1}})] \\
=\ & 1-\alpha.
\end{aligned}$$

■

V. Numerical Results

In this section, we present some numerical results. In this first part, we show the validity of our procedures. Then compared with KN procedure in [15] and KVP, UVP in [12]. To check the statistical validity, we use slippage configuration (SC). In SC, $\mu_1=\mu_2=\cdots=\mu_k-\delta=0$. This is the most difficult configuration for selecting the best because all inferior systems is exactly δ smaller than the best system. On other hand, to check the effectiveness, we run monotone-increasing-means configuration (MIM) as an example configuration. In MIM, $\mu_i=i\delta$, for $i=1,2,\ldots,k$. Also, we examine different configurations of variance: equal variance configuration (EV), increasing-variance configuration (IV), and decreasing-variance configuration (DV). In EV, we check two groups of variance, $\sigma_1^2=\sigma_2^2=\cdots=\sigma_k^2=1$ and $\sigma_1^2=\sigma_2^2=\cdots=\sigma_k^2=10$. In IV, $\sigma_i^2=i$; and in DV, $\sigma_i^2=k-i+1$. Other parameters are set as PCS: $\alpha=0.05$, initial sample size for unknown variance cases: $n_0=16$, indifference-zone parameter: $\delta=1/\sqrt{16}$. system number $k=10$ or $k=25$. For each experiment, we run 10000 macro-replications of each procedure and report the average sample needed (ASN) and PCS.

A. Validity Check

In Section 4, we present the statistical validity of KVAS and UVAS. Table I reports the ASN and PCS of KVAS and KVP procedures. The ASN and PCS of KN, UVP and UVAS are presented in Table II and III. From these three table, we can see that both procedures can select the best system with probability $1-\alpha$ under SC in each group of variance. As mentioned before, SC is the most difficult mean configuration. Table IV, V, VI can prove the validity under MIM configuration. Under MIM, the PCS are all above 0.99, which is much larger than desired PCS. So the procedure needs more samples than necessary to insure the PCS. This is a major shortcoming for indifference-zone assumption, which is discussed in [20], [10].

TABLE I
ASN and PCS of KVAS and KVP under SC.

		KVAS		**KVP**		
σ_i^2	k	ASN	PCS	ASN	PCS	Redution
1	10	803	0.9649	978	0.9661	17.89%
	25	2183	0.9793	2890	0.9648	24.46%
10	10	7976	0.9599	9659	0.9632	17.42%
	25	21485	0.9767	28609	0.9654	24.90%
i	10	3549	0.9590	3894	0.9600	8.86%
	25	22308	0.9599	24543	0.9584	9.11%
$k-i+1$	10	5074	0.9671	5760	0.9686	11.91%
	25	32899	0.9776	39876	0.9720	17.50%

B. The Effectiveness of KVAS and UVAS

Because KVAS and KVPS are for known variance cases, and other three procedures are for unknown variance cases, we discuss the effectiveness of KVAS compared with KVP and compare UVAS with KN and UVP. Table I and IV reports the PCS for SC and MIM configuration, respectively. The reduction is significant for each case. For example, under SC, $\sigma_i^2=10$ and $k=25$, the saving is 24.90%, and under MIM, $\sigma_i^2=10$, and $k=25$, the reduction is 23.35%.

Reduction of UVAS is much larger than KVAS. The saving comes from several sources: 1. using unequal sample size; 2. sampling rule based on total sample size minimization problem; 3. continuously updating variance. By continuously updating variance, UVAS can nearly achieve the performance of known variance procedure KVAS. This is much better than

TABLE II
ASN and PCS of KN and UVAS under SC.

		KN		**UVAS**		
σ_i^2	k	ASN	PCS	ASN	PCS	Redution
1	10	1269	0.9607	801	0.9562	36.88%
	25	3694	0.9678	2135	0.9694	42.20%
10	10	12425	0.9656	7907	0.9623	36.36%
	25	36491	0.9637	21445	0.9721	41.23%
i	10	5216	0.9526	3540	0.9547	32.13%
	25	34011	0.9509	22323	0.9588	34.37%
$k-i+1$	10	8042	0.9685	5067	0.9675	36.99%
	25	55512	0.9791	32729	0.9805	41.04%

TABLE III
ASN and PCS of UVP and UVAS under SC.

		UVP		**UVAS**		
σ_i^2	k	ASN	PCS	ASN	PCS	Redution
1	10	1332	0.9759	801	0.9562	39.86%
	25	4197	0.9860	2135	0.9694	49.13%
10	10	13034	0.9779	7907	0.9623	39.34%
	25	41460	0.9859	21445	0.9721	48.28%
i	10	5259	0.9702	3540	0.9547	32.69%
	25	35856	0.9716	22323	0.9588	37.74%
$k-i+1$	10	7832	0.9802	5067	0.9675	35.30%
	25	59019	0.9864	32729	0.9805	44.54%

TABLE IV

ASN AND PCS OF KVP AND KVAS UNDER MIM.

		KVP		KVAS		
	k	ASN	PCS	ASN	PCS	Reduction
1	10	422	0.9948	356	0.9956	15.64%
	25	672	0.9982	530	0.9980	21.13%
10	10	4169	0.9953	3460	0.9954	17.01%
	25	6680	0.9976	5120	0.9974	23.35%
k	10	3229	0.9947	2822	0.9950	12.60%
	25	13491	0.9981	10798	0.9978	19.96%
$k-i+1$	10	1621	0.9966	1187	0.9940	12.55%
	25	3123	0.9974	2683	0.9978	14.09%

TABLE V

ASN AND PCS OF KN AND UVAS UNDER MIM.

		KN		UVAS		
	k	ASN	PCS	ASN	PCS	Reduction
1	10	557	0.9952	392	0.9936	29.62%
	25	975	0.9983	703	0.9983	27.90%
10	10	5404	0.9950	3486	0.9922	35.49%
	25	8804	0.9979	5186	0.9983	41.09%
i	10	4522	0.9945	2820	0.9950	37.64%
	25	19073	0.9977	10858	0.9984	43.07%
$k-i+1$	10	1440	0.99452	1034	0.9943	28.19%
	25	3875	0.9979	2649	0.9972	31.64%

other non updating unknown variance procedures. Mostly, UVAS can save $30\%-50\%$ compared with KN and UVP. For example, under MIM, $\sigma_i^2 = i$ and $k = 25$, the saving can be 43.07% compared with KN procedure. Compared with UVP, the largest saving is 48.28% for $\sigma_i^2 = 10$, $k = 25$ and under SC.

To summarize, reduction proportion of UVAS is larger than KVAS because compared with other procedures, UVAS is not only an unequal sample size but also a variance-updating procedure.

TABLE VI

ASN AND PCS OF UVP AND UVAS UNDER MIM.

		UVP		UVAS		
	k	ASN	PCS	ASN	PCS	Reduction
1	10	586	0.9969	392	0.9936	33.11%
	25	1029	0.9992	703	0.9983	31.68%
10	10	5615	0.9955	3486	0.9922	37.92%
	25	9593	0.9992	5186	0.9983	45.94%
i	10	4354	0.9959	2820	0.9950	35.23%
	25	19340	0.9994	10858	0.9984	43.86%
$k-i+1$	10	1618	0.9967	1034	0.9943	36.09%
	25	4541	0.9989	2649	0.9972	41.66%

VI. CONCLUSION

In this paper, we present two unequal sample size procedures based on adaptive sampling rule. These procedures deliver desired probability of correct selection and can reduce the total sample size significantly. We also prove the statistical validity for each procedure. For unknown variance procedure UVAS, the sample variance can continuously update, which improve the effectiveness of the procedure.

There are several interesting future work directions. First, it seems possible to implement our idea in indifference-zone procedures. Second, now that the numerical study can support the statistical validity of UVAS, we are interested to extend the proof in finite-time version.

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