Adaptive Fully Sequential Selection Procedures with Linear and Nonlinear Control Variates

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Abstract

A decision-making process often involves selecting the best solution from a finite set of possible alternatives regarding some performance measure, which is known as ranking-and-selection (R&S) when the performance is not explicitly available and can only be estimated by taking samples. Many R&S procedures have been proposed considering different problem formulations. In this paper, we adopt the classical fully sequential indifference-zone (IZ) formulation developed in the statistical literature, and take advantage of the control variates, a well-known variance reduction technique in the simulation literature, to investigate the potential benefits as well as the statistical guarantee by designing a new type of R&S procedure in an adaptive fashion. In particular, we propose a generic adaptive fully sequential procedure that can employ both linear and nonlinear control variates, in which both the control coefficient and sample variance can be sequentially updated as the sampling process progresses. We demonstrate that the proposed procedures provide the desired probability of correct selection in the asymptotic regime as the IZ parameter goes to zero. We then compare the proposed procedures with various existing

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procedures through extensive numerical experiments and two illustrative examples, in which we observe several interesting findings and demonstrate the advantage of our proposed procedures.

Keywords: ranking and selection; adaptive control variates; sample average approximation; stochastic approximation; variance reduction techniques; simulation

1 Introduction

Selecting the best solution with either the largest or smallest mean performance from a finite population is known as ranking-and-selection (R&S) in the statistics and simulation literature, which can be traced back at least to two original papers, i.e., the indifference-zone (IZ) formulation of Bechhofer (1954) and the subset selection formulation of Gupta (1956) in the middle 1950s. Since then, many R&S procedures have been designed to solve different types of R&S problems with different problem formulations.

In the statistics literature, many research works are built on the spirit of statistical hypothesis testing by choosing a proper alternative hypothesis (cf., Kim and Nelson (2006b) and Bechhofer et al. (1995)). For instance, the subset selection procedure in Gupta (1956, 1965) could be viewed as setting the alternative hypothesis as the selected subset containing the true best system, while the one-stage IZ selection procedure in Bechhofer (1954) could be considered as setting the alternative hypothesis as the selected best system is at least δ greater than the second best in terms of the mean, where δ is known as the IZ parameter. Then, the probability of correct decision is closely related to the power calculation of the hypothesis testing, which often can be converted to the calculation of the quantile for normal distribution or the first-passing time for Brownian motion process under the normality assumption. Since the IZ formulation implies a unique best solution, then most R&S procedures, e.g., two-stage selection procedures in Rinott (1978); Nelson et al. (2001) and fully sequential selection procedures in Paulson (1964); Hartmann (1991), are designed under the IZ formulation, in order to provide the targeted statistical guarantee of probability of correct selection (PCS).

In the simulation literature, the development of R&S procedures has been expanded in many ways, considering not only the effectiveness of delivering a valid statistical guarantee, but also the efficiency of using a small number of total samples, which results in two types of approaches in general, i.e., the frequentist approach followed from the existing statistics literature and the Bayesian approach developed from the optimization/dynamic programming literature.

In the frequentist approach, many simulation techniques have been taken into consideration, e.g., common random numbers (CRNs) in fully sequential selection procedure of Kim and Nelson (2001) and other variance reduction techniques (VRTs) in both multi-stage and fully sequential

selection procedures of Tsai and Kuo (2012) and Tsai et al. (2017). Meanwhile, new theoretical building blocks have been established to improve the algorithm efficiency (e.g., constructing variance-dependent Brownian motion process in Hong (2006)), and to solve new types of R&S problems (e.g., conducting asymptotic analysis for steady-state simulation in Kim and Nelson (2006a)), as well as to remove the widely used IZ assumption (e.g., proposing Law of the Iterated Logarithm based continuation boundaries in Fan et al. (2016)). With the fast development of computer technology, parallel computing environments have also been adopted to solve large-scale R&S problems, e.g., the asymptotic parallel selection procedure in Luo et al. (2015) and the good selection procedure in Ni et al. (2017) and Zhong et al. (2022). On the other hand, under the Bayesian approach, instead of delivering a guarantee of correct selection of the best system, Bayesian-type procedures often consider the efficiency as the primary goal to design various (nearly) optimal sample allocation rules under different objective functions, typically by maximizing the posterior PCS or maximizing the value of information, for example, the optimal computing budget allocation (OCBA) family procedures in Chen et al. (2000); Lee et al. (2012); Gao et al. (2017), the value of information procedures in Chick and Inoue (2001); Chick et al. (2010); Qu et al. (2015), and the approximate dynamic control policies in Frazier et al. (2008); Ryzhov (2016); Peng et al. (2018). Notice that the optimality of the sampling efficiency is usually derived in the asymptotic sense as the total number of samples goes to infinity.

In this paper, we focus on employing the VRTs in the frequentist approach, especially considering both the linear and nonlinear control variates (CVs) methods in the design of fully sequential selection procedures. The CV technique is one of the most effective and widely applied VRTs in the simulation literature. The idea of CV is to take advantage of the correlation between the outputs and some selected concomitant random variables (i.e., controls). The existing research on this topic can be divided into two main categories: standard linear CV and nonlinear CV. When applying linear CV estimation, the controls are often chosen among the set of input random variables in the simulated system (with user-specified probability distribution functions). For instance, it is natural to use either the interarrival time or service time as controls to estimate the expected waiting time because of the desired correlations in a queueing system (Law, 2007). It is worthwhile pointing out that the linear CV estimation is easily implemented and computed, basically based on a linear regression technique, however an inappropriate choice of controls may degrade its statistical efficiency. Interested readers may refer to Bauer and Wilson (1992) for a detailed discussion of selecting effective linear controls.

Much research effort for developing a more general CV estimation is needed because many problems naturally admit a nonlinear CV parameterization. The relevant application problems include the computational finance problems (e.g., pricing financial derivatives in Jourdain (2009)

and Staum (2009)), and the problems that involve the simulation of a Markov process (e.g., multiclass queueing networks in Henderson et al. (2003) and Markov chain Monte Carlo examples in Dellaportas and Kontoyiannis (2012)). It requires a comprehensive understanding of the structure of the simulated system to construct nonlinear CV using "approximating martingales", which is initiated by Henderson (1997) that uses an approximating process to define a zero-mean martingale for the original process. It should be noticed that the martingale-type nonlinear CV parameterization needs to be identified from scratch for each specific problem, and the computation of nonlinear CV estimation is not so straightforward as in the linear case. Kim and Henderson (2007) introduce two types of nonlinear CV estimators, which are respectively based on the sample average approximation (SAA) and stochastic approximation (SA) approaches.

The potential benefit of employing linear CV estimation in R&S procedures has been demonstrated in some works in the literature. For instance, Nelson and Staum (2006) propose a two-stage selection procedure while Tsai and Nelson (2010) design fully sequential selection procedures that are effectively combined with the linear CV estimators. Integrated use of linear CV and other VRTs, such as correlation induction, conditional expectation and poststratified sampling, have also been applied into fully sequential selection procedures in Tsai and Kuo (2012) and Tsai et al. (2017). It is worthwhile noticing that all fully sequential selection procedures in the aforementioned VRT-related works employ the linear CV model from which the controlled sum (CS) estimator is derived (cf., Tsai and Nelson (2010)). However, how to incorporate the nonlinear CV estimation into R&S procedure designing, especially into the fully sequential framework, is one interesting problem that has not yet been investigated in the literature, and thus we would like to address in this paper.

Back to the problem of implementing the linear CV in R&S procedure designing, one critical issue is the choice of the control coefficient $\widehat{\boldsymbol{\beta}}$, which is typically set as the optimal value to minimize the variance of the CV estimator. Recall that the CS estimator initially proposed in Tsai and Nelson (2010), which is a variation of ordinary linear CV estimators, has the following two features: (i) it requires a preliminary stage to calculate the control coefficient estimator $\widehat{\boldsymbol{\beta}}$ based on the linear CV model, and (ii) the value of $\widehat{\boldsymbol{\beta}}$ is then fixed when forming the CS estimator used in the subsequent elimination stages. These two requirements are recommended to maintain the finite-time statistical validity of such procedures under normality assumptions. Tsai et al. (2017) show that the normality assumptions might be violated when implementing the linear CV with some other VRTs, and then design fully sequential selection procedures with an asymptotic statistical guarantee. The properties of nonlinear CV estimators, from either SAA or SA approaches, are often derived in the asymptotic sense, which motivates us to investigate the asymptotic statistical validity as employing nonlinear CV in R&S procedures.

As mentioned above, the value of $\widehat{\boldsymbol{\beta}}$ in the CS estimator of Tsai and Nelson (2010) will not be updated even though more and more observations are sequentially collected in the fully sequential selection procedures. In fact, both the CS estimator of Tsai and Nelson (2010) and the SAA estimator of Kim and Henderson (2007) can be viewed as a non-adaptive version because they use independent sets of observations to compute the coefficient $\widehat{\boldsymbol{\beta}}$ and the desired CS estimator of Tsai and Nelson (2010) or the SAA estimator of Kim and Henderson (2007) (see Lapeyre and Lelong (2011) for the definitions of adaptive and non-adaptive estimators). Then, whether it is possible to design an adaptive method that can update the value of $\widehat{\boldsymbol{\beta}}$ in the fully sequential R&S procedures is another interesting problem.

In this paper, we first propose an adaptive fully sequential R&S framework that can tune the linear estimation of control coefficient to improve efficiency as the sampling process progresses. In addition, we aim to adopt the nonlinearly parameterized CV estimation, instead of simply linear CV estimation, in designing adaptive fully sequential selection procedures, which makes it more generally applicable. Since both standard linear CV and martingale-type nonlinear CV have been well studied in the simulation literature, it seems well worthwhile to develop a general and adaptive R&S framework for reaping their benefits.

The rest of the paper is organized as follows. In Section 2, we review the main ideas of linear and nonlinear control variates, and introduce the corresponding notation that will be used throughout the article. In Section 3, we present adaptive fully sequential selection procedures when linear or nonlinear control variates are applied, and discuss the statistical validity and efficiency of these procedures in an asymptotic regime. Numerical results and two practical illustration examples are provided in Sections 4 and 5, respectively. The paper ends with some concluding remarks in Section 6. The proofs and the details of the benchmark procedures are contained in the Appendix.

2 Preliminaries

Suppose that there are k independent simulation systems with unknown mean performance measure θ_i for each system design $i = 1, 2, \ldots, k$. Let Y_{ij} denote the simulation output from the jth replication of system i. The ordinary sample-mean-based R&S procedures assume that

$$Y_{ij} = \theta_i + \eta_{ij},$$

where η_{ij} is independently and identically distributed (i.i.d.) normal random variable $\mathrm{N}(0, \sigma_i^2)$ with unknown σ_i^2 for all i and j. The standard estimator of θ_i across n simulation replications is the

sample mean estimator

$$\bar{Y}_i(n) = \frac{1}{n}\sum_{j=1}^{n} Y_{ij},$$

which is unbiased and has variance σ_i^2/n.

2.1 Linear Control Variates

We first review the linear CV estimator. Let $\mathbf{C}_{ij}$ represent the function of simulation inputs from the jth replication of system i. The $q_i \times 1$ vector $\mathbf{C}_{ij}$ is called the *control* and is assumed to have a known mean vector $\boldsymbol{\mu}_i$. Let $\mathbf{C}_i(n)$ denote the $n \times q_i$ matrix whose jth row is $\mathbf{C}_{ij}^{\mathsf{T}}$. Let $\mathbf{Y}_i(n) = (Y_{i1}, \ldots, Y_{in})^{\mathsf{T}}$. We consider the following linear CV estimator of θ_i:

$$\widehat{\theta}_{\mathrm{CV}}(i,n) = \bar{Y}_i(n) - \left(\bar{\mathbf{C}}_i(n) - \boldsymbol{\mu}_i\right)^{\mathsf{T}} \widehat{\boldsymbol{\beta}}_i(n), \tag{1}$$

where $\bar{Y}_i(n) = \frac{1}{n}\sum_{j=1}^{n} Y_{ij}$ and $\bar{\mathbf{C}}_i(n) = \frac{1}{n}\sum_{j=1}^{n} \mathbf{C}_{ij}$ are the sample means of the outputs and controls, respectively, and

$$\widehat{\boldsymbol{\beta}}_i(n) = \mathbf{S}_{\mathbf{C}_i}^{-1}(n)\mathbf{S}_{\mathbf{C}_i Y_i}(n), \tag{2}$$

where $\mathbf{S}_{\mathbf{C}_i}(n) = (n-1)^{-1}\left(\mathbf{C}_i(n)^{\mathsf{T}}\mathbf{C}_i(n) - n\bar{\mathbf{C}}_i(n)\bar{\mathbf{C}}_i(n)^{\mathsf{T}}\right)$ is the sample variance-covariance matrix of $\mathbf{C}_{ij}$, and $\mathbf{S}_{\mathbf{C}_i Y_i}(n) = (n-1)^{-1}\left(\mathbf{C}_i(n)^{\mathsf{T}}\mathbf{Y}_i(n) - n\bar{\mathbf{C}}_i(n)\bar{Y}_i(n)\right)$ is the sample covariance vector between $\mathbf{C}_{ij}$ and Y_{ij}. See Remark 1 for finite-sample properties of $\widehat{\theta}_{\mathrm{CV}}(i,n)$ under the normality assumption of $(Y_{ij}, \mathbf{C}_{ij})$.

Without the normality assumption, Nelson (1990) derives some nice properties of the CV estimator in the asymptotic sense, which are the weak law of large numbers (WLLN) and central limit theorem (CLT) for $\widehat{\theta}_{\mathrm{CV}}(i,n)$ (cf., Theorem 3 in Nelson (1990)). That is, as $n \to \infty$, $\widehat{\theta}_{\mathrm{CV}}(i,n) \xrightarrow{\mathrm{p}} \theta_i$ and $\sqrt{n}\left(\widehat{\theta}_{\mathrm{CV}}(i,n) - \theta_i\right) \Rightarrow N(0, \xi_i^2)$, where $\xrightarrow{\mathrm{p}}$ denotes convergence in probability and $\Rightarrow$ denotes convergence in distribution, and $\xi_i^2 = (1 - R_i^2)\sigma_i^2$ with $R_i^2 = \mathrm{Cov}\left[Y_{ij}, \mathbf{C}_{ij}\right]\mathrm{Var}\left[\mathbf{C}_{ij}\right]^{-1}\mathrm{Cov}\left[\mathbf{C}_{ij}, Y_{ij}\right]/\sigma_i^2$ is the square of the multiple correlation coefficient between Y_{ij} and $\mathbf{C}_{ij}$. The asymptotic results allow us to derive the PCS guarantee as incorporating the adaptively updated CV estimators into designing the fully sequential R&S procedures.

The standard linear CV estimator is statistically more efficient than the ordinary sample mean estimator (as long as $R_i^2 > q_i/(n-2)$), and has been successfully adopted in designing R&S procedures. For instance, Tsai and Nelson (2010) propose a variation of standard linear CV estimators, called CS estimators, that can be appropriately incorporated into fully sequential procedures. It requires to collect a preliminary-stage sample $\{(Y_{ij}, \mathbf{C}_{ij}), j = 1, 2, \ldots, m_0\}$ to compute $\widehat{\boldsymbol{\beta}}_i(m_0)$ for each system $i = 1, 2, \ldots, k$ (based on Equation (2)), and a first-stage sample $\{(Y_{ij}, \mathbf{C}_{ij}), j = m_0+1, m_0+2, \ldots, m_0+n_0\}$ to compute the the controlled sample mean $\bar{Y}_i[m_0, n_0]$

and the controlled sample variance $S_i^2[m_0, n_0]$ as follows,

$$\begin{aligned}
\bar{Y}_i\,[m_0, n_0] &= \frac{1}{n_0}\sum_{j=m_0+1}^{m_0+n_0}\Big[Y_{ij} - (\mathbf{C}_{ij} - \boldsymbol{\mu}_i)^{\mathrm{T}}\,\widehat{\boldsymbol{\beta}}_i(m_0)\Big], \\
S_i^2\,[m_0, n_0] &= \frac{1}{n_0 - 1}\sum_{j=m_0+1}^{m_0+n_0}\Big[Y_{ij} - (\mathbf{C}_{ij} - \boldsymbol{\mu}_i)^{\mathrm{T}}\,\widehat{\boldsymbol{\beta}}_i(m_0) - \bar{Y}_i[m_0, n_0]\Big]^2.
\end{aligned}$$

Under the normality assumption described in Remark 1, it can be shown that $\mathrm{E}\left[\bar{Y}_i[m_0, n_0]\right] = \theta_i$ and $\mathrm{Var}\left[\bar{Y}_i[m_0, n_0]\right] = \left(\frac{m_0-2}{m_0-q_i-2}\right)\frac{\xi_i^2}{n_0}$. Then, the values of control coefficient $\widehat{\boldsymbol{\beta}}_i$ and variance estimator S_i^2 are both fixed in subsequent stages of the elimination process. The disadvantages of using CS estimators include: (i) the preliminary-stage observations are wasted because they are collected before the elimination process is initiated; and (ii) the loss ratio contains only the preliminary-stage sample size m_0 instead of the overall sample size, which implies that its statistical efficiency is inferior to that of the standard CV estimators.

Remark 1. *In most of the existing literature, e.g., Nelson and Staum (2006) and Tsai and Nelson (2010), it is often assumed that* $\{(Y_{ij}, \mathbf{C}_{ij}), j = 1, 2, \ldots\}$ *are i.i.d.* $(q_i + 1) \times 1$ *normal random vectors. Under the assumption of multivariate normality,* Y_{ij} *can be described as follows:*

$$Y_{ij} = \theta_i + (\mathbf{C}_{ij} - \boldsymbol{\mu}_i)^{\mathrm{T}}\boldsymbol{\beta}_i^* + \epsilon_{ij},$$

where the multiplier $\boldsymbol{\beta}_i^* = \mathrm{Var}\,[\mathbf{C}_{ij}]^{-1}\,\mathrm{Cov}\,[\mathbf{C}_{ij}, Y_{ij}]$*, which is unknown and is called the* optimal control coefficient *in terms of minimizing the variance of the CV estimator. The residual terms* $\{\epsilon_{ij}, j = 1, 2, \ldots\}$ *are i.i.d.* $\mathrm{N}(0, \xi_i^2)$ *random variables where* $\xi_i^2 = (1 - R_i^2)\sigma_i^2$*. Then, we know that* $\mathrm{E}[\widehat{\theta}_{CV}(i, n)] = \theta_i$ *and* $\mathrm{Var}[\widehat{\theta}_{CV}(i, n)] = \left(\frac{n-2}{n-q_i-2}\right)\frac{\xi_i^2}{n}$*, where the term* $\left(\frac{n-2}{n-q_i-2}\right)$ *is known as the loss ratio.*

2.2 Nonlinear Control Variates

The nonlinear CV estimator of θ_i considered in the paper is defined as follows:

$$\widehat{\theta}_{\mathrm{NCV}}(i, n) = \bar{Y}_i(n) - \frac{1}{n}\sum_{j=1}^{n} f(\mathbf{C}_{ij}, \widehat{\boldsymbol{\beta}}_i), \tag{3}$$

where $\mathbf{C}_{ij}$ is a $q_i \times 1$ vector of control, $\widehat{\boldsymbol{\beta}}_i$ is a $p_i \times 1$ vector of control coefficient estimators, and $f(\mathbf{C}_{ij}, \widehat{\boldsymbol{\beta}}_i)$ could be any nonlinear function of $\mathbf{C}_{ij}$ and $\widehat{\boldsymbol{\beta}}_i$. Unlike the linear CV estimation in Equation (1), a nonlinear parameterization could result in different numbers of dimensionality of the vectors $\mathbf{C}_{ij}$ and $\widehat{\boldsymbol{\beta}}_i$. For instance, when $f(\mathbf{C}_{ij}, \widehat{\boldsymbol{\beta}}_i)$ is represented as $\widehat{\beta}_{i1}\mathrm{C}_{ij1}^{\widehat{\beta}_{i2}} + \widehat{\beta}_{i3}\mathrm{C}_{ij2} + \widehat{\beta}_{i4}$, then

the dimensionality of $\mathbf{C}_{ij}$ and $\widehat{\boldsymbol{\beta}}_i$ is $q_i = 2$ and $p_i = 4$, respectively.

For nonlinear CV estimation, we do not explicitly assume any intrinsic relationship between Y_{ij} and $\mathbf{C}_{ij}$, and do not assume the control mean is known. However, we need to assume that an appropriate nonlinearly parameterized function $f(\mathbf{C}_{ij}, \widehat{\boldsymbol{\beta}}_i)$ is available and its expectation equals to zero for any value of $\widehat{\boldsymbol{\beta}}_i$, in order to make $\widehat{\theta}_{\text{NCV}}$ unbiased. This assumption is often true when using the martingale-type control variates in simulation (cf., Ashford and Beale (1989); Henderson and Glynn (2002); Kim and Henderson (2007)). Moreover, we cannot expect to find a closed form expression for $\widehat{\boldsymbol{\beta}}_i$ as in the linear case (i.e., Equation (2)) when the CV parameterization is nonlinear. Fortunately, Kim and Henderson (2007) show that, under some pathwise differentiability and moment conditions, the variance of nonlinear CV estimator becomes a differentiable function with respect to the parameter $\widehat{\boldsymbol{\beta}}_i$, and therefore stochastic optimization approaches can be applied to search for the optimal value of $\widehat{\boldsymbol{\beta}}_i$. In this paper, we adopt the two types of nonlinear CV estimators, based on the SAA and SA stochastic optimization approaches derived in Kim and Henderson (2007), to investigate the possibility and potential issues as combining into fully sequential procedures. We next briefly introduce the SAA and SA algorithms, i.e., Algorithms 1 and 2, for computing the nonlinear CV estimators, denoted as $\widehat{\theta}_{\text{SAA}}$ and $\widehat{\theta}_{\text{SA}}$, respectively.

The SAA-based nonlinear CV estimator is derived by a two-phase approach. In the first phase, Algorithm 1 estimates $\widehat{\boldsymbol{\beta}}_i(m)$ by computing the first-order critical point for the minimization problem of the sample variance for $\{Y_{ij} - f(\mathbf{C}_{ij}, \boldsymbol{\beta}_i), j = 1, 2, \ldots, m\}$ for each system i.[1] In the second phase, it uses $\widehat{\boldsymbol{\beta}}_i(m)$ in place of $\widehat{\boldsymbol{\beta}}_i$ and then computes $\widehat{\theta}_{\text{NCV}}(i, n)$ via Equation (3) based on newly collected n observations $\{(Y_{ij}, \mathbf{C}_{ij}), j = m+1, \ldots, m+n\}$.

Algorithm 1 (The SAA algorithm for computing $\widehat{\theta}_{\text{SAA}}$)**.**

First-phase: Choose a positive integer $m \geq 2$. Generate the i.i.d. observations $\{(Y_{ij}, \mathbf{C}_{ij}), j = 1, 2, \ldots, m\}$. Find $\widehat{\boldsymbol{\beta}}_i(m)$, a first-order critical point of the variance minimization problem $\min_{\boldsymbol{\beta}_i} \text{Var}(\boldsymbol{\beta}_i, m)$, where $\text{Var}(\boldsymbol{\beta}_i, m)$ is defined as follows,

$$\text{Var}(\boldsymbol{\beta}_i, m) = \frac{1}{m-1}\left[\sum_{j=1}^{m}\left(\widehat{\theta}_{ij}\right)^2 - \frac{1}{m}\left(\sum_{j=1}^{m}\widehat{\theta}_{ij}\right)^2\right],$$

with $\widehat{\theta}_{ij} = Y_{ij} - f(\mathbf{C}_{ij}, \boldsymbol{\beta}_i)$ for any fixed $\boldsymbol{\beta}_i$.

Second-phase: Generate the i.i.d. observations $\{(Y_{ij}, \mathbf{C}_{ij}), j = m+1, m+2, \ldots, m+n\}$, independent of the first-phase observations. Then, compute $\widehat{\theta}_{\text{SAA}}(i, n) = n^{-1}\sum_{j=m+1}^{m+n}\left[Y_{ij} - f(\mathbf{C}_{ij}, \widehat{\boldsymbol{\beta}}_i(m))\right]$ as the desired estimator.

[1] Notice that, unlike the linear case of obtaining an optimal value of $\widehat{\boldsymbol{\beta}}_i(m)$, the best we can hope for the nonlinear case here is to obtain one of the first-order critical points of the minimization problem.

Remark 2. *It is worthwhile pointing out that when $f(\mathbf{C}_{ij}, \boldsymbol{\beta}_i)$ is a linear function, i.e., $f(\mathbf{C}_{ij}, \boldsymbol{\beta}_i) = (\mathbf{C}_{ij} - \boldsymbol{\mu}_i)^{\mathrm{T}} \boldsymbol{\beta}_i$, the first-order critical point $\widehat{\boldsymbol{\beta}}_i(m)$ obtained in Algorithm 1 is indeed the global optimal value, which has the closed-form expression as in Equation (2), i.e., $\widehat{\boldsymbol{\beta}}_i(m) = \mathbf{S}^{-1}_{\mathbf{C}_i}(m)\mathbf{S}_{\mathbf{C}_i Y_i}(m)$.*

Different from the two-phase approach of SAA algorithm, the SA-based algorithm works in a recursive way which is analogous to the steepest-descent gradient search method in deterministic optimization, except that the gradient is stochastic and needs to be estimated. We introduce additional notation w to represent the iteration index. At iteration w, the sample mean and the gradient estimator of the sample variance, with respect to m observations $\{Y^w_{ij} - f(\mathbf{C}^w_{ij}, \widehat{\boldsymbol{\beta}}^{w-1}_i), j = 1, 2, \ldots, m\}$, are computed conditioned on the $\widehat{\boldsymbol{\beta}}^{w-1}_i$ estimated at iteration $(w-1)$, for each system i. Suppose that $\widehat{\boldsymbol{\beta}}_i$ takes the value in the set $\mathbf{B}$, i.e., $\widehat{\boldsymbol{\beta}}_i \in \mathbf{B} \subset \mathbb{R}^p$. Then, the estimate $\widehat{\boldsymbol{\beta}}^{w-1}_i$ is updated to $\widehat{\boldsymbol{\beta}}^w_i$ as follows: $\widehat{\boldsymbol{\beta}}^w_i = \Pi_{\mathbf{B}}(\widehat{\boldsymbol{\beta}}^{w-1}_i - a_w g_{w-1}(\widehat{\boldsymbol{\beta}}^{w-1}_i))$, where $\Pi_{\mathbf{B}}$ denotes a projection of points outside $\mathbf{B}$ back into $\mathbf{B}$, $a_w > 0$ is the step size satisfying $\sum_{w=1}^{\infty} a_w = \infty$ and $\sum_{w=1}^{\infty} a^2_w < \infty$, and $g_{w-1}(\widehat{\boldsymbol{\beta}}^{w-1}_i)$ is the gradient estimator from iteration w.

Algorithm 2 (The SA algorithm for computing $\widehat{\theta}_{\mathrm{SA}}$)**.**

Initialization: Choose $\widehat{\boldsymbol{\beta}}^0_i$, for $i = 1, 2, \ldots, k$, and a sequence of positive real numbers $\{a_w, w = 1, 2, \ldots, n\}$, where n is the total number of iterations.

For-Loop: For $w = 1$ to n

Generate the i.i.d. observations $\{(Y^w_{ij}, \mathbf{C}^w_{ij}), j = 1, 2, \ldots, m\}$, where $m \geq 2$, and compute

$$
\begin{aligned}
\mathrm{A}^w_i(\widehat{\boldsymbol{\beta}}^{w-1}_i) &= \frac{1}{m}\sum_{j=1}^{m}\left[Y^w_{ij} - f(\mathbf{C}^w_{ij}, \widehat{\boldsymbol{\beta}}^{w-1}_i)\right], \\
g_{w-1}(\widehat{\boldsymbol{\beta}}^{w-1}_i) &= \frac{-2}{m-1}\sum_{j=1}^{m}\Bigg\{\Big[Y^w_{ij} - f(\mathbf{C}^w_{ij}, \widehat{\boldsymbol{\beta}}^{w-1}_i) - \mathrm{A}^w_i(\widehat{\boldsymbol{\beta}}^{w-1}_i)\Big] \\
&\qquad \nabla_{\widehat{\boldsymbol{\beta}}_i}\Big[f(\mathbf{C}^w_{ij}, \widehat{\boldsymbol{\beta}}_i) - \frac{1}{m}\sum_{l=1}^{m} f(\mathbf{C}^w_{il}, \widehat{\boldsymbol{\beta}}_i)\Big]\Big|_{\widehat{\boldsymbol{\beta}}_i = \widehat{\boldsymbol{\beta}}^{w-1}_i}\Bigg\}, \\
\widehat{\boldsymbol{\beta}}^w_i &= \Pi_{\mathbf{B}}\big(\widehat{\boldsymbol{\beta}}^{w-1}_i - a_w g_{w-1}(\widehat{\boldsymbol{\beta}}^{w-1}_i)\big).
\end{aligned}
$$

Next w.

After-the-Loop. Then, compute $\widehat{\theta}_{\mathrm{SA}}(i, n) = n^{-1}\sum_{w=1}^{n} \mathrm{A}^w_i(\widehat{\boldsymbol{\beta}}^{w-1}_i)$ as the desired estimator.

Kim and Henderson (2007) also show some nice properties regarding the strong law of large numbers (SLLN) and CLT for $\widehat{\theta}_{\mathrm{SAA}}$ and $\widehat{\theta}_{\mathrm{SA}}$ obtained in Algorithms 1 and 2. To achieve this goal, Kim and Henderson (2007) require three assumptions regarding the unbiased property and

differentiability conditions for $\widehat{\theta}_{\text{SAA}}$ and $\widehat{\theta}_{\text{SA}}$, and two additional assumptions specifically for $\widehat{\theta}_{\text{SA}}$, as well as some independence conditions, most of which are also required to prove the limiting properties for $\widehat{\theta}_{\text{NCV}}$ in Equation (3).[2] In particular, Kim and Henderson (2007) demonstrate that, as $n \to \infty$, $\widehat{\theta}_{\text{SAA}}(i,n) \xrightarrow{\text{a.s.}} \theta_i$ and $\widehat{\theta}_{\text{SA}}(i,n) \xrightarrow{\text{a.s.}} \theta_i$, where $\xrightarrow{\text{a.s.}}$ denotes convergence with probability one, and $\sqrt{n}(\widehat{\theta}_{\text{SAA}}(i,n) - \theta_i) \Rightarrow v^{1/2}(\widehat{\boldsymbol{\beta}}_i(m))N(0,1)$ and $\sqrt{mn}(\widehat{\theta}_{\text{SA}}(i,n) - \theta_i) \Rightarrow v^{1/2}(\boldsymbol{\beta}_i^*)N(0,1)$, where $v(\boldsymbol{\beta}_i) = \text{Var}\left[Y_{ij} - f(\mathbf{C}_{ij}, \boldsymbol{\beta}_i)\right]$.

3 The Procedures

In this section, we present fully sequential selection procedures that allow either linear or nonlinear CV estimation and adaptively update the $\widehat{\boldsymbol{\beta}}$ estimator as the elimination process progresses. We demonstrate the statistical validity of the proposed procedures in a meaningful asymptotic regime as the IZ parameter δ goes to zero.

Following the traditional IZ formulation, we assume that $\theta_1 - \delta \geq \theta_2 \geq \ldots \geq \theta_k$, where system 1 with the largest mean performance measure is the true best and the mean difference between the best and second best is greater than or equal to the IZ parameter δ. Our goal is to design a fully sequential procedure that can provide the asymptotic PCS level $1-\alpha$, that is,

$$\liminf_{\delta \to 0} \ \Pr\left\{\text{select system } 1 | \theta_1 - \theta_2 \geq \delta\right\} \geq 1 - \alpha.$$

Notice that if the smallest mean performance is desired, then multiply each observation Y_{ij} by -1 before implementing the procedure.

3.1 The Generic Adaptive Fully Sequential Procedure with Control Variates

We first present a generic framework from which the adaptive fully sequential procedures with linear and nonlinear control variates can be derived in Sections 3.2 and 3.3.

The procedure works as follows. We first set the initial control coefficient estimator to an arbitrary value as desired. We then take first-stage n_0 observations to calculate the initial variance estimators for each system $i = 1, 2, \ldots, k$, and conduct the comparisons for elimination decisions with the first-stage samples. Then, at each of the following rounds, we take one observation for each system still in contention in a round-robin order, and perform necessary statistics updating and elimination comparisons. Continuing doing so, it stops until there is only one system left. Notice that, after the first stage, the control coefficient estimator $\widehat{\boldsymbol{\beta}}$ for each system is calculated

[2] Interested readers may refer to Proposition 3 in Appendix B for a similar result for $\widehat{\theta}_{\text{SAA}}$ obtained in our proposed procedure.

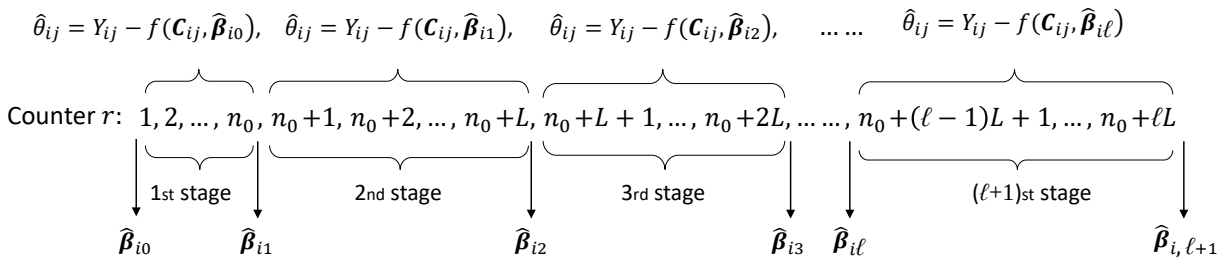

Figure 1: The sequence for $\widehat{\boldsymbol{\beta}}_{i\ell}$ updating and $\widehat{\theta}_{ij}$ calculation.

based on all observations collected up to the updating points, which could be consecutive at the end of each round or every batch of several rounds.

For notational simplicity, we let L denote the batching size of rounds, that is the number of observations between two consecutive updating points. Notice that when $L = 1$, we then update $\widehat{\boldsymbol{\beta}}$ at each round when one additional observation is obtained.[3] Let r be the sample counter for observations collected from each system and let ℓ be the stage counter for $\widehat{\boldsymbol{\beta}}$ updating points. Then, $\ell = 0$ when $r = 0$, $\ell = 1$ when $1 \leq r \leq n_0$ and $\ell = \lceil (r - n_0)/L \rceil + 1$ when $r > n_0$, where $\lceil x \rceil$ denotes the smallest integer that is greater than or equal to x. At the end of the ℓth stage ($\ell \geq 1$), where the sample counter $r = n_0 + (\ell - 1)L$, the coefficient estimator $\widehat{\boldsymbol{\beta}}$ for system i, denoted by $\widehat{\boldsymbol{\beta}}_{i\ell}$, is computed based on all r observations using either the linear or nonlinear CV methods described in Sections 2.1 and 2.2. For the jth controlled observation of system i, which is in the following $(\ell + 1)$st stage, we define accordingly $\widehat{\theta}_{ij} = Y_{ij} - f\big(\mathbf{C}_{ij}, \widehat{\boldsymbol{\beta}}_{i\ell}\big)$, where $f\big(\mathbf{C}_{ij}, \widehat{\boldsymbol{\beta}}_{i\ell}\big)$ could be a linear or nonlinear function of $\mathbf{C}_{ij}$ and $\widehat{\boldsymbol{\beta}}_{i\ell}$. In other words, this updating scheme allows $\widehat{\boldsymbol{\beta}}_{i\ell}$ to be applied only to the observations collected in the $(\ell + 1)$st stage (i.e., jth observation where $j = \{n_0 + (\ell - 1)L + 1, n_0 + (\ell - 1)L + 2, \ldots, n_0 + \ell L\}$), as shown in Figure 1. In the current version of procedure, for each system i we choose $\widehat{\boldsymbol{\beta}}_{i0}$ such that $f\big(\mathbf{C}_{ij}, \widehat{\boldsymbol{\beta}}_{i0}\big) = 0$ for the observations in the first stage, i.e., $j = 1, 2, \ldots, n_0$. In general, the value of $\widehat{\boldsymbol{\beta}}_{i0}$ could be arbitrary as desired, for example, setting as the value estimated by prior knowledge or experience. We call the fully sequential procedure using the previously defined $\widehat{\theta}_{ij}$ as $\mathcal{AFS}$.

Procedure 1 (The Generic Adaptive Fully Sequential Procedure ($\mathcal{AFS}$))**.**

Step 0. Setup: Select the PCS level $1 - \alpha \in (1/k, 1)$, IZ parameter $\delta > 0$, first-stage sample size $n_0 \geq 2$, initial value of $\widehat{\boldsymbol{\beta}}_{i0}$ such that $f\big(\mathbf{C}_{ij}, \widehat{\boldsymbol{\beta}}_{i0}\big) = 0$ for all $i = 1, 2, \ldots, k$,[4] and $\widehat{\boldsymbol{\beta}}$ updating frequency $L \geq 1$ (or $L \geq 2$). Let $a = -\log[2\alpha/(k-1)]$.

[3]Note that, when using SA-based approach, it requires that $L \geq 2$ in order to estimate the gradient at least in the first iteration.

[4]For the linear case, we can simply set $\widehat{\boldsymbol{\beta}}_{i0} = 0$ for all $i = 1, 2, \ldots, k$. For the nonlinear case, the setting of $\widehat{\boldsymbol{\beta}}_{i0}$ could be problem-dependent.

Step 1. Initialization: Let $I = \{1, 2, \ldots, k\}$ be the set of systems still in contention. Obtain n_0 observations $\{(Y_{ij}, \mathbf{C}_{ij}), j = 1, 2, \ldots, n_0\}$ from each system $i \in I$. Set the stage counter $\ell = 1$, and compute the jth controlled observation of system i as

$$\widehat{\theta}_{ij} = Y_{ij} - f\big(\mathbf{C}_{ij}, \widehat{\boldsymbol{\beta}}_{i,\ell-1}\big)$$

for all $i \in I$ and $j = 1, 2, \ldots, n_0$. Set the sample counter $r = n_0$.

Step 2. Updates: Compute the sample mean and sample variance of the first r controlled observations for all systems $i \in I$ as follows,

$$\bar{\theta}_i(r) = \frac{1}{r}\sum_{j=1}^{r}\widehat{\theta}_{ij}, \quad \text{and} \quad S_i^2(r) = \frac{1}{r-1}\Big[\sum_{j=1}^{r}\big(\widehat{\theta}_{ij}\big)^2 - \frac{1}{r}\Big(\sum_{j=1}^{r}\widehat{\theta}_{ij}\Big)^2\Big]. \tag{4}$$

Step 3. Elimination: Set $I^{\text{old}} = I$. Let

$$I = I^{\text{old}} \backslash \Bigg\{ i \in I^{\text{old}} : \bar{\theta}_i(r) - \bar{\theta}_h(r) < \min\left\{0, -\frac{a}{\delta}\cdot\frac{\big[S_i^2(r) + S_h^2(r)\big]}{r} + \frac{\delta}{2}\right\}, \text{ for some } h \in I^{\text{old}} \text{ and } h \neq i \Bigg\},$$

where $A \setminus B = \{x : x \in A \text{ and } x \notin B\}$.

Step 4. Stopping Rule: If $|I| = 1$, then stop and select the system whose index is in I as the best. Otherwise,

(i). Updating $\widehat{\boldsymbol{\beta}}$: If $r = n_0 + (\ell - 1)L$, then set $\widehat{\boldsymbol{\beta}}_{i\ell} = \widehat{\boldsymbol{\beta}}_i(r)$ for each system $i \in I$, where $\widehat{\boldsymbol{\beta}}_i(r)$ is calculated based on linear or nonlinear CV estimation, and let $\ell = \ell + 1$.

(ii). Generating Sample: Let $r = r + 1$. Take the rth sample $(Y_{ir}, \mathbf{C}_{ir})$ from system $i \in I$, and obtain the rth controlled observation of system $i \in I$ as

$$\widehat{\theta}_{ir} = Y_{ir} - f\big(\mathbf{C}_{ir}, \widehat{\boldsymbol{\beta}}_{i,\ell-1}\big),$$

and go to **Step 2**.

Notice that in the elimination process in **Step 3** of $\mathcal{AFS}$, we estimate the variance of the observation difference using the marginal variance estimators for each system, which is reasonable because the CRN technique is not used. In the following subsections, we will incorporate both the linear and nonlinear $\widehat{\boldsymbol{\beta}}$ updating methods into **Step 4(i)** of $\mathcal{AFS}$, and demonstrate their statistical validity through the asymptotic analysis, which, as a byproduct, allows us to update the sample

variances sequentially as shown in the procedure. The asymptotic regime used here essentially assumes that the first-stage sample size $n_0 = n_0(\delta)$ is a function of δ, satisfying $n_0 \to \infty$ and $\delta n_0 \to 0$ as $\delta \to 0$. It is also worthwhile pointing out that this asymptotic regime along with variance updating mechanism has also been used in Luo et al. (2015) and Tsai et al. (2017).

3.2 The Adaptive Fully Sequential Procedure with Linear Control Variates

When using the linear CV estimation in Section 2.1, at the end of ℓth stage, $\ell = 1, 2, \ldots$, we update $\widehat{\boldsymbol{\beta}}_{i\ell} = \widehat{\boldsymbol{\beta}}_i(r) = \mathbf{S}^{-1}_{\mathbf{C}_i}(r)\mathbf{S}_{\mathbf{C}_i Y_i}(r)$ according to Equation (2), with $r = n_0 + (\ell - 1)L$. Then, we increase the stage counter $\ell = \ell + 1$ and obtain $\widehat{\theta}_{ij} = Y_{ij} - (\mathbf{C}_{ij} - \boldsymbol{\mu}_i)^{\mathrm{T}} \widehat{\boldsymbol{\beta}}_{i,\ell-1}$ as the jth controlled observation of system $i \in I$ in this stage.. It should be noted that, for $\mathcal{AFS}$ with linear CV estimation, the estimator $\widehat{\theta}_{ij}$ is unbiased because the expectation operator is linear and $\widehat{\boldsymbol{\beta}}_{i,\ell-1}$ is estimated from observations in the previous stages which is independent with $\mathbf{C}_{ij}$ at the current stage. The updating rule is described in detail in Algorithm 3 as follows.

Algorithm 3 (The linear method for updating $\widehat{\boldsymbol{\beta}}$)**.**

Updating Rule: At stage ℓ when sample size $r = n_0 + (\ell - 1)L$, we obtain $\widehat{\boldsymbol{\beta}}_{i\ell} = \widehat{\boldsymbol{\beta}}_i(r)$ as follows,

$$\widehat{\boldsymbol{\beta}}_i(r) = \mathbf{S}^{-1}_{\mathbf{C}_i}(r)\mathbf{S}_{\mathbf{C}_i Y_i}(r),$$

where $\mathbf{S}_{\mathbf{C}_i}(r)$ is the sample variance-covariance matrix of $\mathbf{C}_{ij}$, and $\mathbf{S}_{\mathbf{C}_i Y_i}(r)$ is the sample covariance vector between $\mathbf{C}_{ij}$ and Y_{ij}, based on observations $\{(Y_{ij}, \mathbf{C}_{ij}), j = 1, 2, \ldots, r\}$.

Remark 3. *Since the estimator $\widehat{\boldsymbol{\beta}}$ has an explicit formula in the case of linear CV, the updates can be calculated using only prior estimates and new observations, and then the storage of previous individual observations is unnecessary. For instance, when $q_i = 1$ we have $\widehat{\beta}_i(r) = S_{C_i Y_i}(r)/S^2_{C_i}(r)$, where $S_{C_i Y_i}(r)$ is the sample covariance between C_{ij} and Y_{ij}, and $S^2_{C_i}(r)$ is the sample variance of C_{ij}, based on observations $\{(Y_{ij}, C_{ij}), j = 1, 2, \ldots, r\}$. For the next update of $\widehat{\beta}_i$ (i.e., when additional observations $\{(Y_{ij}, C_{ij}), j = r+1, r+2, \ldots, r+L\}$ are available), we first define $\widetilde{Y}_i(r, L) = [\sqrt{r} \cdot \bar{Y}_i(r) + \sqrt{r+L} \cdot \bar{Y}_i(r+L)]/[\sqrt{r} + \sqrt{r+L}]$ and similarly define $\widetilde{C}_i(r, L)$. We can then compute $\widehat{\beta}_i(r+L) = [S_{C_i Y_i}(r) + S_{C_i Y_i}(*)]/[S^2_{C_i}(r) + S^2_{C_i}(*)]$, where $S_{C_i Y_i}(*) = \sum_{j=r+1}^{r+L}[C_{ij} - \widetilde{C}_i(r, L)][Y_{ij} - \widetilde{Y}_i(r, L)]$ and $S^2_{C_i}(*) = \sum_{j=r+1}^{r+L}[C_{ij} - \widetilde{C}_i(r, L)]^2$. Interested readers may refer to Escobar and Moser (1993) and Klotz (1995) for more detailed discussion on updating formulae for simple or multiple linear regression without keeping all previous observations.*

Since the updated mechanism of $\widehat{\boldsymbol{\beta}}$ leads to a complex correlation structure of controlled observations, it appears difficult to show a finite-time statistical validity of the proposed procedures. More specifically, the controlled observations in the same ℓth stage remain i.i.d. (as conditioned on

the same $\widehat{\boldsymbol{\beta}}_{i,\ell-1}$), but the observations collected in different stages are not i.i.d. anymore, because the computation of $\widehat{\boldsymbol{\beta}}_{i\ell}$ is based on samples in the previous stages. Fortunately, we can show the asymptotic statistical validity in the regime as the IZ parameter $\delta \to 0$. The main result for the linear CV updating scheme is summarized in Theorem 1.

Theorem 1. *Let $(Y_{ij}, \mathbf{C}_{ij})$ be the jth simulation outputs and controls from system i, which is assumed to be independent for different i and j, where $j = 1, 2, \ldots$ and $i = 1, 2, \ldots, k$. Let θ_i be the mean performance measure of system i (i.e., $\theta_i = \mathrm{E}[Y_{ij}]$) satisfying the IZ formulation, which is assumed as $\theta_1 - \delta \geq \theta_2 \geq \ldots \geq \theta_k$, where δ is the predetermined IZ parameter. Let the first-stage sample size $n_0 = n_0(\delta)$ be a function of δ, satisfying that $n_0(\delta) \geq \max_{i=1,\ldots,k} q_i + 3$, and that $n_0 \to \infty$ and $\delta n_0 \to 0$ as $\delta \to 0$. Then, with the coefficient $\widehat{\boldsymbol{\beta}}$ updating rule in Algorithm 3, the $\mathcal{AFS}$ procedure provides a correct selection (i.e., selecting system 1 as the best) with a probability at least $1 - \alpha$ as $\delta \to 0$.*

Notice that the result in Theorem 1 still holds if replacing the linear updating rule in Algorithm 3 by the nonlinear SAA updating rule in Algorithm 4, under some mild conditions, which will be introduced in the following subsection.

3.3 The Adaptive Fully Sequential Procedure with Nonlinear Control Variates

For the nonlinear CV case, we first consider employing the SAA nonlinear CV estimator of Algorithm 1 for $\widehat{\boldsymbol{\beta}}$ updating scheme in the generic procedure $\mathcal{AFS}$, which is called $\mathcal{AFS}$-$\mathcal{SAA}$ and is described in detail as follows.

In **Step 4(i)** of $\mathcal{AFS}$-$\mathcal{SAA}$, at the end of ℓth stage, $\ell = 1, 2, \ldots$, the control coefficient estimator $\widehat{\boldsymbol{\beta}}_{i\ell} = \widehat{\boldsymbol{\beta}}_i(r)$ with $r = n_0 + (\ell - 1)L$, is computed in the same manner as shown in the first-phase of Algorithm 1 introduced in Section 2.2. In particular, we use all the outcomes collected up to the current updating point, i.e., $\{(Y_{ij}, \mathbf{C}_{ij}), j = 1, 2, \ldots, r\}$, and the given nonlinear function $f\big(\mathbf{C}_{ij}, \boldsymbol{\beta}_i\big)$ to obtain the associated $\mathrm{Var}(\boldsymbol{\beta}_i, r)$. By finding the first-order critical point for the SAA minimization problem $\min_{\boldsymbol{\beta}_i} \mathrm{Var}(\boldsymbol{\beta}_i, r)$, we obtain the optimal value of $\widehat{\boldsymbol{\beta}}_i(r)$, which is the desired value of $\widehat{\boldsymbol{\beta}}_{i\ell}$. It is worthwhile pointing out that, as known the exact form of function $f\big(\mathbf{C}_{ij}, \boldsymbol{\beta}_i\big)$, we may not necessarily need to record every outcome of $\{(Y_{ij}, \mathbf{C}_{ij}), j = 1, 2, \ldots, r\}$, but to record some particular statistics in order to compute the terms $\sum_{j=1}^{r} \widehat{\theta}_{ij}$ and $\sum_{j=1}^{r} \big(\widehat{\theta}_{ij}\big)^2$ in the expression of $\mathrm{Var}(\boldsymbol{\beta}_i, r)$. In the following Algorithm 4, we describe the updating rule used in **Step 4(i)** of $\mathcal{AFS}$-$\mathcal{SAA}$ in detail.

Algorithm 4 (The SAA nonlinear method for updating $\widehat{\boldsymbol{\beta}}$)**.**

Updating Rule: At stage ℓ with sample size $r = n_0 + (\ell - 1)L$, we obtain $\widehat{\boldsymbol{\beta}}_{i\ell} = \widehat{\boldsymbol{\beta}}_i(r)$ by finding a first-order critical point of the SAA optimization problem

$$\min_{\boldsymbol{\beta}_i} \mathrm{Var}(\boldsymbol{\beta}_i, r) = \min_{\boldsymbol{\beta}_i} \frac{1}{r-1}\Big[\sum_{j=1}^{r}\big(\widehat{\theta}_{ij}\big)^2 - \frac{1}{r}\Big(\sum_{j=1}^{r}\widehat{\theta}_{ij}\Big)^2\Big]$$

where $\widehat{\theta}_{ij} = Y_{ij} - f(\mathbf{C}_{ij}, \boldsymbol{\beta}_i)$ for any fixed $\boldsymbol{\beta}_i$, based on all observations $\big\{(Y_{ij}, \mathbf{C}_{ij}), j = 1, 2, \ldots, r\big\}$.

The SAA-based CV estimator $\widehat{\theta}_{\mathrm{SAA}}$ used in Kim and Henderson (2007) implements only two phases of sampling, and the independence between these two phases is required to achieve the SLLN and CLT. In other words, as shown in Algorithm 1, the m observations used in constructing $\widehat{\boldsymbol{\beta}}_i(m)$ are independent of those n observations used in computing $\widehat{\theta}_{\mathrm{SAA}}(i, n)$. By contrast, our proposed adaptive fully sequential procedure involves multiple stages of sampling, and the samples employed for computing $\widehat{\boldsymbol{\beta}}_i(r)$ and $\bar{\theta}_i(r)$ are not independent. In addition, $\widehat{\boldsymbol{\beta}}_i(r)$ is a random variable that depends on sampling in previous stages (instead of a deterministic approximation of the optimal value $\boldsymbol{\beta}_i^*$), therefore the ordinary SLLN and CLT cannot be immediately applied. In Appendix B, we derive the desired SLLN and CLT of $\bar{\theta}_i(r)$ used in $\mathcal{AFS}$-$\mathcal{SAA}$, for which we need more assumptions than those in Kim and Henderson (2007).

For the ease of reading, we need to define some notation. Let $\boldsymbol{\beta}_i^*$ be one of the first-order critical points for the variance minimization problem of $\min_{\boldsymbol{\beta}_i} \mathrm{Var}\left[Y_{ij} - f(\mathbf{C}_{ij}, \boldsymbol{\beta}_i)\right]$. Let H denote the support of the probability distribution of $(Y_{ij}, \mathbf{C}_{ij})$, i.e., H is the smallest closed set such that $\Pr\{(Y_{ij}, \mathbf{C}_{ij}) \in H\} = 1$. Let H_2 be the set of all $\mathbf{c}$ values that appear in H, i.e., $H_2 = \{\mathbf{c} : \exists y \text{ such that } (y, \mathbf{c}) \in H\}$. Then, four assumptions are presented as follows.

Assumption 1. The random variable Y_{ij} is square integrable. Also, for all $\boldsymbol{\beta}_i \in \mathscr{U}$, $\mathrm{E}[f(\mathbf{C}_{ij}, \boldsymbol{\beta}_i)] = 0$ and $\mathrm{E}[f^2(\mathbf{C}_{ij}, \boldsymbol{\beta}_i)] < \infty$, where $\mathscr{U}$ is a bounded open set containing $\mathbf{B}$.

Assumption 2. The parameter set $\mathbf{B}$ is compact. For all $\mathbf{c} \in H_2$, the real-valued function $f(\mathbf{c}, \cdot)$ is continuously differentiable on $\mathscr{U}$.

Assumption 3. For all $\mathbf{c} \in H_2$, $f(\mathbf{c}, \cdot)$ is Lipschitz on $\mathscr{U}$, i.e., there exists $W(\mathbf{c}) > 0$ such that for all $\boldsymbol{\beta}_i$, $\boldsymbol{\beta}_i' \in \mathscr{U}$,

$$|f(\mathbf{c}, \boldsymbol{\beta}_i) - f(\mathbf{c}, \boldsymbol{\beta}_i')| \le W(\mathbf{c})\|\boldsymbol{\beta}_i - \boldsymbol{\beta}_i'\|,$$

where $\|\cdot\|$ is a metric on $\mathbb{R}^p$. Therefore,

$$\sup_{\boldsymbol{\beta}_i \in \mathscr{U}} \left|\frac{\partial f(\mathbf{c}, \boldsymbol{\beta}_i)}{\partial\,(\boldsymbol{\beta}_i)_d}\right| \le W(\mathbf{c}),$$

where $\partial(\boldsymbol{\beta}_i)_d$ means taking the partial derivative with respect to the dth component of the vector $\boldsymbol{\beta}_i$ for all $\mathbf{c} \in H_2$ and all $d = 1, 2, \ldots, p$. Moreover, we assume that $\mathrm{E}\big[\,(W(\mathbf{C}_{ij}))^2\,\big] < \infty$.

Assumption 4. Suppose that a local optimization method (e.g., Newton method) with initial guess $\widetilde{\boldsymbol{\beta}}_{i0}$ is applied to solve the SAA optimization problem in Algorithm 4. Then, there exists $\epsilon > 0$ such that $\widetilde{\boldsymbol{\beta}}_{i0} \in \mathcal{B}(\epsilon, \boldsymbol{\beta}_i^*)$, where $\mathcal{B}(\epsilon, \boldsymbol{\beta}_i^*)$ is an ϵ-neighborhood of $\boldsymbol{\beta}_i^*$.[5]

Assumption 1 is used to make the nonlinear CV estimator to be unbiased (i.e., $\mathrm{E}[Y_{ij} - f(\mathbf{C}_{ij}, \boldsymbol{\beta}_i)] = \theta_i$). Assumption 2 implies that for each $(Y_{ij}, \mathbf{C}_{ij}) \in H$, the estimator $Y_{ij} - f(\mathbf{C}_{ij}, \cdot)$ is continuously differentiable on $\mathscr{U}$. Moreover, if $f(\mathbf{c}, \cdot)$ is continuously differentiable on the entire space $\mathbb{R}^p$ or on an open set containing the closure of $\mathscr{U}$, then Assumption 3 holds immediately. Under Assumptions 1-3, it can be shown that the variance function $\mathrm{Var}[Y_{ij} - f(\mathbf{C}_{ij}, \boldsymbol{\beta}_i)]$, which is a function of $\boldsymbol{\beta}_i$, is continuously differentiable in $\boldsymbol{\beta}_i \in \mathbf{B}$ (cf., Proposition 3.1 of Kim and Henderson (2007)). Assumption 4 is used to provide the convergence of the updated control coefficient parameter $\widehat{\boldsymbol{\beta}}_i$ to a constant $\boldsymbol{\beta}^*$. Note that Assumptions 1-3 are often used in the stochastic optimization literature and we bring them directly from Kim and Henderson (2007), while Assumption 4 is commonly used in the numerical optimization literature (see Griewank and Osborne (1983) and Nocedal and Wright (1999) for more applications).

We are now ready to present our main results for $\mathcal{AFS}$-$\mathcal{SAA}$ in Theorem 2.

Theorem 2. *Let $(Y_{ij}, \mathbf{C}_{ij})$ be the jth simulation outputs and controls from system i, which is assumed to be independent for different i and j, where $j = 1, 2, \ldots$ and $i = 1, 2, \ldots, k$. Let θ_i be the mean performance measure of system i (i.e., $\theta_i = \mathrm{E}[Y_{ij}]$) satisfying the IZ formulation, which is assumed as $\theta_1 - \delta \geq \theta_2 \geq \ldots \geq \theta_k$, where δ is the predetermined IZ parameter. Suppose that Assumptions 1–4 hold. Let the first-stage sample size $n_0 = n_0(\delta)$ be a function of δ, satisfying that $n_0(\delta) \geq 2$ and that $n_0 \to \infty$ and $\delta n_0 \to 0$ as $\delta \to 0$. Then, with the coefficient $\widehat{\boldsymbol{\beta}}$ updating rule in Algorithm 4, the $\mathcal{AFS}$-$\mathcal{SAA}$ procedure provides a correct selection (i.e., selecting system 1 as the best) with a probability at least $1 - \alpha$ as $\delta \to 0$.*

Different from Theorem 1, in which the linear CV estimator implies the convergence of the coefficient $\widehat{\boldsymbol{\beta}}$ to the optimal $\boldsymbol{\beta}^*$, resulting a variance reduction and thus a total sample size reduction of the $\mathcal{AFS}$, the nonlinear coefficient $\widehat{\boldsymbol{\beta}}$ updating rule in Theorem 2 converges only to one of the first-order critical points $\boldsymbol{\beta}^*$, therefore may not guarantee a variance reduction or a total sample size reduction of $\mathcal{AFS}$-$\mathcal{SAA}$ (which is also true for $\mathcal{AFS}$-$\mathcal{SA}$). However, in practice, we observe that both $\mathcal{AFS}$-$\mathcal{SAA}$ and $\mathcal{AFS}$-$\mathcal{SA}$ can provide significant total sample size savings based on our simulation studies in Sections 4 and 5.

[5] When a local optimization method is applied to implement the SA approach in Algorithm 5, $\widetilde{\boldsymbol{\beta}}_{i0} = \widehat{\boldsymbol{\beta}}_i^0$, which is equal to $\widehat{\boldsymbol{\beta}}_{i0}$, so we just assume that $\widehat{\boldsymbol{\beta}}_{i0} \in \mathcal{B}(\epsilon, \boldsymbol{\beta}_i^*)$.

When deriving the CLT of $\bar{\theta}_i(r)$ used in $\mathcal{AFS}$-$\mathcal{SAA}$, we propose two ways. One is based on $\delta \to 0$ implying that $n_0 \to \infty$, which is in accord with the setting of Theorem 1. Furthermore, we also find that even though n_0 does not converge to infinity (i.e., n_0 is finite), the CLT of $\bar{\theta}_i(r)$ still holds under some dependent conditions, and we keep this result in Appendix B for interested readers.

It is also worthwhile pointing out that, the convergence of $\widehat{\boldsymbol{\beta}}$ to a constant (i.e., Assumption 4) allows us to obtain a constant limiting variance estimator, which is critical to properly scale the discrete-time process. Based on that, we can prove Theorem 2 following exactly the same spirit as proving Theorem 1.

We next consider employing the SA nonlinear CV estimator of Algorithm 2 for $\widehat{\boldsymbol{\beta}}$ updating scheme in the generic procedure $\mathcal{AFS}$, which is called $\mathcal{AFS}$-$\mathcal{SA}$. Recall that, for system i, the SA algorithm implements n iterations, each consists of m observations, resulting mn total number of samples, to recursively update $\widehat{\boldsymbol{\beta}}$ and then return $\widehat{\theta}_{\rm SA}(i,n)$ as in Algorithm 2. Therefore, we need to assume that both n_0 and L can be divided by m in order to recursively apply SA approach in each stage.

In **Step 4(i)** of $\mathcal{AFS}$-$\mathcal{SA}$, we compute the estimator $\widehat{\boldsymbol{\beta}}_{i\ell} = \widehat{\boldsymbol{\beta}}_i^w$ where $w = \frac{n_0}{m}$ when $\ell = 1$ and $w = \frac{L}{m}$ when $\ell = 2, 3, \ldots$. In particular, we need to pre-specify the starting solution $\widehat{\boldsymbol{\beta}}_i^0 = \widehat{\boldsymbol{\beta}}_{i0}$ at the first stage, while at other stage $\ell \geq 2$, we can simply set $\widehat{\boldsymbol{\beta}}_i^0 = \widehat{\boldsymbol{\beta}}_{i,\ell-1}$ as obtained from the previous stage. As applying the SA method in each stage, we may use the same sequence of step sizes $\{a_w, w = 1, 2, \ldots\}$. In the following Algorithm 5, we describe the updating rule used in **Step 4(i)** of $\mathcal{AFS}$-$\mathcal{SA}$ in detail.

Algorithm 5 (The SA nonlinear method for updating $\widehat{\boldsymbol{\beta}}$)**.**

Updating Rule: At stage $\ell = 1$, let $\widehat{\boldsymbol{\beta}}_i^0 = \widehat{\boldsymbol{\beta}}_{i0}$,

- **For-Loop**: For $w = 1$ to $\frac{n_0}{m}$, compute

$$\begin{aligned}
\mathrm{A}_i^w(\widehat{\boldsymbol{\beta}}_i^{w-1}) &= \frac{1}{m}\sum_{j=(w-1)m+1}^{wm}\big[Y_{ij} - f(\mathbf{C}_{ij}, \widehat{\boldsymbol{\beta}}_i^{w-1})\big], \\
g_{w-1}(\widehat{\boldsymbol{\beta}}_i^{w-1}) &= \frac{-2}{m-1}\sum_{j=(w-1)m+1}^{wm}\Bigg\{\Big[Y_{ij} - f(\mathbf{C}_{ij}, \widehat{\boldsymbol{\beta}}_i^{w-1}) - \mathrm{A}_i^w(\widehat{\boldsymbol{\beta}}_i^{w-1})\Big] \\
&\qquad \nabla_{\widehat{\boldsymbol{\beta}}_i}\Big[f(\mathbf{C}_{ij}, \widehat{\boldsymbol{\beta}}_i) - \frac{1}{m}\sum_{l=(w-1)m+1}^{wm} f(\mathbf{C}_{il}, \widehat{\boldsymbol{\beta}}_i)\Big]\Big|_{\widehat{\boldsymbol{\beta}}_i = \widehat{\boldsymbol{\beta}}_i^{w-1}}\Bigg\}, \\
\widehat{\boldsymbol{\beta}}_i^w &= \Pi_{\mathbf{B}}\big(\widehat{\boldsymbol{\beta}}_i^{w-1} - a_w g_{w-1}(\widehat{\boldsymbol{\beta}}_i^{w-1})\big).
\end{aligned}$$

  Next w.

- **After-the-Loop**. Then, let $\widehat{\boldsymbol{\beta}}_{i1} = \widehat{\boldsymbol{\beta}}_i^{\frac{n_0}{m}}$.

At other stages $\ell = 2, 3, \ldots$, let $\widehat{\boldsymbol{\beta}}_i^0 = \widehat{\boldsymbol{\beta}}_{i,\ell-1}$,

- **For-Loop**: For $w = 1$ to $\frac{L}{m}$, compute

$$\begin{aligned}
\mathrm{A}_i^w(\widehat{\boldsymbol{\beta}}_i^{w-1}) &= \frac{1}{m} \sum_{j=n_0+(\ell-2)L+(w-1)m+1}^{n_0+(\ell-2)L+wm} \big[Y_{ij} - f(\mathbf{C}_{ij}, \widehat{\boldsymbol{\beta}}_i^{w-1})\big], \\
g_{w-1}(\widehat{\boldsymbol{\beta}}_i^{w-1}) &= \frac{-2}{m-1} \sum_{j=n_0+(\ell-2)L+(w-1)m+1}^{n_0+(\ell-2)L+wm} \Big\{ \Big[Y_{ij} - f(\mathbf{C}_{ij}, \widehat{\boldsymbol{\beta}}_i^{w-1}) - \mathrm{A}_i^w(\widehat{\boldsymbol{\beta}}_i^{w-1})\Big] \\
&\qquad \nabla_{\widehat{\boldsymbol{\beta}}_i} \Big[f(\mathbf{C}_{ij}, \widehat{\boldsymbol{\beta}}_i) - \frac{1}{m} \sum_{l=n_0+(\ell-2)L+(w-1)m+1}^{n_0+(\ell-2)L+wm} f(\mathbf{C}_{il}, \widehat{\boldsymbol{\beta}}_i) \Big] \Big|_{\widehat{\boldsymbol{\beta}}_i = \widehat{\boldsymbol{\beta}}_i^{w-1}} \Big\}, \\
\widehat{\boldsymbol{\beta}}_i^w &= \Pi_{\mathbf{B}}\big(\widehat{\boldsymbol{\beta}}_i^{w-1} - a_w g_{w-1}(\widehat{\boldsymbol{\beta}}_i^{w-1})\big).
\end{aligned}$$

  Next w.

- **After-the-Loop**. Then, let $\widehat{\boldsymbol{\beta}}_{i\ell} = \widehat{\boldsymbol{\beta}}_i^{\frac{L}{m}}$.

Remark 4. *Kim and Henderson (2007) provide a SLLN and CLT of the SA-based nonlinear CV estimator whose assumptions can be verified by our Assumptions 1-3. Interested readers may refer to Kim and Henderson (2007) for more details of the corresponding SLLN and CLT. Note that, with additional assumption (i.e., Assumption 4) that the initial guess of a local optimization method is in a neighborhood of the first-order stationary point $\boldsymbol{\beta}_i^*$, we have $\widehat{\boldsymbol{\beta}}_{i1} \to \boldsymbol{\beta}_i^*$, as $n_0 \to \infty$. Then, the initial $\widehat{\boldsymbol{\beta}}_i^0$ at stage $\ell \geq 2$ will start from $\boldsymbol{\beta}_i^*$, implying that $\widehat{\boldsymbol{\beta}}_{i\ell} = \boldsymbol{\beta}_i^*$, which allows us to demonstrate the asymptotic statistical validity of $\mathcal{AFS}$-$\mathcal{SA}$ in the same manner as $\mathcal{AFS}$-$\mathcal{SAA}$. Thus, we omit the theoretical result for $\mathcal{AFS}$-$\mathcal{SA}$.*

Notice that rather than computing the $\widehat{\theta}_{\mathrm{SAA}}$ and $\widehat{\theta}_{\mathrm{SA}}$ in Algorithms 1 and 2, the purpose of Algorithms 4 and 5 are used to compute the coefficient $\widehat{\boldsymbol{\beta}}_{i\ell}$. With these nonlinear $\widehat{\boldsymbol{\beta}}$ updating schemes, we are able to solve more general R&S problems, which will be demonstrated in the following.

4 Numerical Experiments

In this section, we perform an extensive numerical evaluation to compare the proposed adaptive procedures with existing fully sequential procedures with and without the CV technique. In particular, we implement two ordinary fully sequential procedures, that is, the $\mathcal{KN}$ procedure of Kim and Nelson (2001), which employs the original sample means (i.e., without CV), and the $\mathcal{TN}$ procedure

of Tsai and Nelson (2010), which employs the CS estimators (i.e., with CV). The details of both $\mathcal{KN}$ and $\mathcal{TN}$ are presented in Appendix C.

We also implement an alternative fully sequential procedure, denoted as $\mathcal{AFS}+$, where the newly updated $\widehat{\boldsymbol{\beta}}_{i\ell}$ will be applied to all r observations collected in all stages,[6] instead of being applied to only observations collected in the $(\ell+1)$st stage as in $\mathcal{AFS}$. $\mathcal{AFS}+$ serves as a heuristic benchmark for $\mathcal{AFS}$, which implies a lower bound of the required number of simulated observations that $\mathcal{AFS}$ can achieve. That is because as applying the most recent $\widehat{\boldsymbol{\beta}}$, which tends to be more accurate than previously estimated $\widehat{\boldsymbol{\beta}}$ as using more samples, to all on-hand observations, $\mathcal{AFS}+$ could sharpen the comparison and thus eliminate the inferior systems earlier than $\mathcal{AFS}$ does. However, the disadvantages of $\mathcal{AFS}+$ are clear. First, we cannot prove its statistical validity even in the asymptotic regime. Second, it requires a much larger computational overhead since it needs to retain all individual observations from all surviving systems and recompute the point and variance estimators whenever $\widehat{\boldsymbol{\beta}}$ is updated. This could require substantial data storage capacity when the number of competing systems is large. By contrast, $\mathcal{AFS}$ only needs to maintain the summary statistics ($\sum_{j=1}^{r}\widehat{\theta}_{ij}$, $\sum_{j=1}^{r}(\widehat{\theta}_{ij})^2$) and perform summation (for the observations in the current stage) to update the point and variance estimators.

The system outputs are represented by various configurations of k normal distributions, among which system 1 is always the best system, i.e., has the largest true mean in all cases. We drop the subscript j for convenience. Let Y_i be a simulation observation from system i, for $i = 1, 2, \ldots, k$. For tractability and simplicity, we assume that each system has one control (i.e., $q_i = 1$ for all i) for both linear and nonlinear cases.

For the linear case, we assume that the observation can be represented as

$$Y_i = \theta_i + \beta_i(C_i - \mu_i) + \epsilon_i,$$

where $\{\epsilon_i,\ i = 1, 2, \ldots, k\}$ are $\mathrm{N}(0, \sigma_\epsilon^2)$ random variables. The input random variables $\{C_i,\ i = 1, 2, \ldots, k\}$ are $\mathrm{N}(0, \sigma_c^2)$ random variables (i.e., $\mu_i = 0$) and independent of $\{\epsilon_i,\ i = 1, 2, \ldots, k\}$. We also set $\beta_i = 1$ for each system $i = 1, 2, \ldots, k$. Therefore, $\{Y_i,\ i = 1, 2, \ldots, k\}$ are distributed as $\mathrm{N}(\theta_i, \sigma_y^2)$ random variables, where $\sigma_y^2 = \sigma_c^2 + \sigma_\epsilon^2$. The squared correlation coefficient between Y_i and C_i is $R_{Y,C}^2 = \sigma_c^2/(\sigma_c^2 + \sigma_\epsilon^2)$ for each system $i = 1, 2, \ldots, k$.

For the nonlinear case, we adopt the polynomial CV model introduced by Nelson (1987) by

[6] That is, in **Step 1** and **Step 4(ii)** of $\mathcal{AFS}+$, we need to redefine and recompute the jth controlled observation from any surviving system i as: $\widehat{\theta}_{ij} = Y_{ij} - f\big(\mathbf{C}_{ij}, \widehat{\boldsymbol{\beta}}_{i\ell}\big)$, for all $j = 1, 2, \ldots, r$, where r is the current sample counter satisfying $n_0 + (\ell - 1)L + 1 \le r \le n_0 + \ell L$.

assuming that the observation can be represented as the following nonlinear function,

$$Y_i = \theta_i + \beta_{i1}(C_i - \mu_i) + \beta_{i2}(C_i - \mu_i)^3 + \epsilon_i,$$

where $\boldsymbol{\beta}_i = (\beta_{i1}, \beta_{i2})^{\mathrm{T}}$ and $\{\epsilon_i,\ i = 1, 2, \ldots, k\}$ are $\mathrm{N}(0, \sigma_\epsilon^2)$ random variables. We assume that $\mu_i = 0$ and therefore $\{C_i,\ i = 1, 2, \ldots, k\}$ are $\mathrm{N}(0, \sigma_c^2)$ random variables and independent of $\{\epsilon_i,\ i = 1, 2, \ldots, k\}$. It should be noticed that $\mathrm{E}[(C_i - \mu_i)] = \mathrm{E}[(C_i - \mu_i)^3] = 0$ because C_i is assumed to be normally distributed. We also set $\beta_{i1} = \beta_{i2} = 1$ for each system $i = 1, 2, \ldots, k$. Therefore, $\{Y_i,\ i = 1, 2, \ldots, k\}$ are random variables with mean θ_i and variance $\sigma_y^2 = \sigma_c^2 + 6\sigma_c^4 + 15\sigma_c^6 + \sigma_\epsilon^2$. Notice that the assumed nonlinear function is intrinsically linear,[7] but we choose to use the nonlinear CV estimation for evaluating the efficiency of both $\mathcal{AFS}$-$\mathcal{SAA}$ and $\mathcal{AFS}$-$\mathcal{SA}$. This intrinsically linear function also allows us to conveniently specify the value of $R^2_{Y,\mathbf{C}}$, where $\mathbf{C} = ((C_i - \mu_i), (C_i - \mu_i)^3)$. After some tedious algebra, we can obtain that $R^2_{Y,\mathbf{C}} = (3375\sigma_c^{18} + 1350\sigma_c^{16} + 405\sigma_c^{14} + 144\sigma_c^{12} + 27\sigma_c^{10} + 6\sigma_c^8 + \sigma_c^6)/\sigma_y^2$. According to the notation defined in Section 2.2, a natural nonlinear CV estimator is $\widehat{\theta}_i = Y_i - \widehat{\beta}_{i1}(C_i - \mu_i) - \widehat{\beta}_{i2}(C_i - \mu_i)^3$, which means that $f\big(C_i, \widehat{\boldsymbol{\beta}}_i\big) = \widehat{\beta}_{i1}(C_i - \mu_i) + \widehat{\beta}_{i2}(C_i - \mu_i)^3$.

We compare the performance of each procedure on different variations of the systems, with examining factors including the practically significant difference δ, the number of systems k, the $\widehat{\boldsymbol{\beta}}$ updating frequency L, the configurations of the system means θ_i, and the squared correlation coefficient between outputs and controls R^2. Notice that CRNs are not employed. The configurations, the experimental design, and the results are described below.

4.1 Configurations and Experimental Design

We examine the slippage configuration (SC) of the true means of the systems in which θ_1 is set to exactly δ, while $\theta_2 = \theta_3 = \cdots = \theta_k = 0$. This is the least favorable scenario in order to achieve the requested PCS, because all the inferior systems are very close to the best system. We choose $\delta = \sigma_y/\sqrt{n_0}$; therefore, the IZ parameter can be interpreted as one standard deviation of the first-stage sample mean. To examine the efficiency of these procedures in eliminating noncompetitive systems, the configuration of monotone-decreasing means (MDM) is also used. In the MDM configuration, the means of systems are determined according to the following formula: $\theta_1 = \delta$ and $\theta_i = \theta_1 - (i - 1)(\delta/2)$, for $i = 2, 3, \ldots, k$.

The number of systems simulated in each experiment is varied with $k = 10, 30, 50, 100$. In all experiments, we set the output variance $\sigma_y^2 = 2$, the nominal PCS $1 - \alpha = 0.95$, and the first-stage sample size $n_0 = 20$, for all procedures. In addition, for the $\mathcal{TN}$-type procedure, which requires a

[7] Interested readers may refer to Chapter 13 of Kutner et al. (2005) for the reasons why the linear regression may not be appropriate for a nonlinear response function that is intrinsically linear.

preliminary-stage to estimate the control coefficient, we then set the preliminary-stage sample size $m_0 = 10$ and 20 (or 30) when employing the linear and nonlinear CV estimation, respectively. These algorithm parameter settings are based on the guidelines provided in Tsai and Nelson (2010) and Tsai and Kuo (2012). For the proposed adaptive fully sequential procedures, we examine different $\widehat{\boldsymbol{\beta}}$ updating frequency $L = \{1, 5, 20\}$ for $\mathcal{AFS}$, $\mathcal{AFS}+$ and $\mathcal{AFS}$-$\mathcal{SAA}$, and $L = \{2, 6, 20\}$ (with $m = 2$) for $\mathcal{AFS}$-$\mathcal{SA}$ (since $\mathcal{AFS}$-$\mathcal{SA}$ requires $L > 1$). For all procedures with CV estimation, we set the initial solution $\widehat{\boldsymbol{\beta}}_{i0} = \mathbf{0}$. When using the SA algorithm, we set the step-size sequence $a_w = 0.6/w$, and the range of the parameter set $\mathbf{B}$ equals to $[-10, 10]$.

For each configuration, 500 trials (i.e., complete macro-repetitions) of each procedure are performed to compare the performance measures, including the estimated PCS and the average number of simulated observations per system (ANS). To simplify the presentation, we round the values of PCS and ANS to the nearest hundredth and integer number, respectively.

4.2 Results for Procedures with Linear Control Variates

In Table 1, we investigate the effect of different levels of correlations (i.e., $R^2_{Y,C}$) and $\widehat{\boldsymbol{\beta}}$ updating frequency (i.e., L) on the performance of both $\mathcal{AFS}$ and $\mathcal{AFS}+$ with linear CV estimation, and compare them to $\mathcal{KN}$ and $\mathcal{TN}$ in the same slippage configuration.

From the results in Table 1, we have several interesting findings. First, $\mathcal{AFS}$ performs quite well, in terms of ANS, compared with $\mathcal{KN}$ and $\mathcal{TN}$, in all tested cases with various parameter settings. To be more specific, we notice that $\mathcal{TN}$ obtains at least a 20% reduction in ANS, compared to $\mathcal{KN}$, as long as $R^2_{Y,C}$ is greater than or equal to 0.3.[8] The adaptive procedure $\mathcal{AFS}$ can further improve the statistical efficiency with at least a 30% reduction in ANS, compared to $\mathcal{TN}$, which indicates the importance of adaptively updating $\widehat{\boldsymbol{\beta}}$. Second, even though we demonstrate the needs of updating $\widehat{\boldsymbol{\beta}}$, frequent updating is not necessary. For instance, there is no significant improvement of saving ANS when updating $\widehat{\boldsymbol{\beta}}$ every round as obtaining one additional observation from all surviving systems (i.e., $L = 1$), compared with doing that every 20 rounds (i.e., $L = 20$). Third, the experimental results also indicate that, compared to $\mathcal{AFS}$, $\mathcal{AFS}+$ achieves a little improvement in ANS,[9] which in other words means that the performance of $\mathcal{AFS}$ has nearly achieved the lower bound in terms of total sample sizes. Hence, it is likely that the additional improvement in ANS by using $\mathcal{AFS}+$ would be negated by its increased computational overhead as compared to $\mathcal{AFS}$. Last but not least, all procedures, including the heuristic $\mathcal{AFS}+$, achieve the nominal PCS level 0.95 in all configurations, which validates the statistical guarantee of these procedures.

[8] The larger value of the correlation $R^2_{Y,C}$ is, the more benefit then $\mathcal{TN}$ and $\mathcal{AFS}$ could provide.

[9] In the paper, we only present the results of $\mathcal{AFS}+$ under the setting of $R^2_{Y,C} = 0.3$ for the reason of brevity. We have obtained similar results in other parameter settings.

Table 1: Performance measures for $\mathcal{AFS}$ and $\mathcal{AFS}+$ (with linear CV estimation) in comparison with $\mathcal{KN}$ and $\mathcal{TN}$ in the SC when $m_0 = 10$, $n_0 = 20$, and $1 - \alpha = 0.95$.

	$R^2_{Y,C}$	L	$k = 10$		$k = 30$		$k = 50$		$k = 100$	
			PCS	ANS	PCS	ANS	PCS	ANS	PCS	ANS
$\mathcal{KN}$			0.97	316	0.99	401	0.98	438	0.99	508
$\mathcal{TN}$	0.3		0.98	256	0.98	324	0.99	365	0.99	407
$\mathcal{AFS}$	0.3	20	0.97	173	0.97	209	0.98	226	0.97	250
	0.3	5	0.95	173	0.96	206	0.98	225	0.97	248
	0.3	1	0.98	172	0.97	206	0.97	221	0.97	245
$\mathcal{AFS}+$	0.3	20	0.95	166	0.95	204	0.98	221	0.98	243
	0.3	5	0.97	163	0.98	200	0.99	217	0.99	240
	0.3	1	0.98	161	0.98	197	0.97	213	0.97	240
$\mathcal{TN}$	0.5		0.98	190	0.98	237	0.99	266	0.99	294
$\mathcal{AFS}$	0.5	20	0.95	122	0.96	148	0.96	160	0.97	178
	0.5	5	0.95	122	0.95	146	0.96	158	0.98	176
	0.5	1	0.95	120	0.98	144	0.97	158	0.97	175
$\mathcal{TN}$	0.7		0.98	116	0.98	146	0.99	161	0.98	183
$\mathcal{AFS}$	0.7	20	0.97	73	0.96	87	0.96	94	0.99	104
	0.7	5	0.96	72	0.96	87	0.96	94	0.99	104
	0.7	1	0.95	72	0.95	87	0.96	93	0.96	103

Table 2: Performance measures for $\mathcal{AFS}$ (with linear CV estimation) in comparison with $\mathcal{KN}$ and $\mathcal{TN}$ in the MDM configuration when $m_0 = 10$, $n_0 = 20$, and $1 - \alpha = 0.95$.

	$R^2_{Y,C}$	L	$k = 10$		$k = 30$		$k = 50$		$k = 100$	
			PCS	ANS	PCS	ANS	PCS	ANS	PCS	ANS
$\mathcal{KN}$			1	211	1	133	1	103	1	70
$\mathcal{TN}$	0.3		0.99	182	1	118	1	92	1	68
$\mathcal{AFS}$	0.3	20	0.99	115	1	70	1	55	1	40
	0.3	5	1	115	1	70	1	54	1	40
	0.3	1	1	115	1	69	1	54	1	40
$\mathcal{TN}$	0.5		1	133	1	87	1	70	1	55
$\mathcal{AFS}$	0.5	20	1	83	1	52	1	42	1	33
	0.5	5	1	83	1	52	1	42	1	33
	0.5	1	1	82	1	51	1	42	1	33
$\mathcal{TN}$	0.7		1	82	1	58	1	51	1	43
$\mathcal{AFS}$	0.7	20	1	50	1	35	1	31	1	27
	0.7	5	1	49	1	35	1	31	1	27
	0.7	1	1	49	1	35	1	31	1	27

In Table 2, we compare the performance of $\mathcal{AFS}$ with $\mathcal{KN}$ and $\mathcal{TN}$ in the MDM configuration.[10] Notice that the empirical PCS performance for all procedures is closer to 1, which is not so surprising since many of the inferior systems become far away from the true best in the MDM configuration. It is worthwhile pointing out that the advantages of the proposed adaptive procedure in terms of ANS as aforementioned still hold. However, there is less incremental benefit when increasing the frequency of updating $\widehat{\boldsymbol{\beta}}$, which reinforces our conclusion that adaptively updating $\widehat{\boldsymbol{\beta}}$ is needed but not necessary to be very frequent.

4.3 Results for Procedures with Nonlinear Control Variates

Since there are no existing R&S procedures designed to handle nonlinear CV, we first modify $\mathcal{TN}$ procedure in Tsai and Nelson (2010) to be suitable for the nonlinear CV estimation with SAA and SA approaches, denoted as $\mathcal{TN}$-$\mathcal{SAA}$ and $\mathcal{TN}$-$\mathcal{SA}$, respectively.

For $\mathcal{TN}$-$\mathcal{SAA}$, we compute the estimator $\widehat{\boldsymbol{\beta}}_i(m_0)$ as in the first-phase of the SAA algorithm (i.e., Algorithm 1) in Section 2.2, where m_0 serves as the preliminary-stage sample size. Then, the values of $\widehat{\boldsymbol{\beta}}_i(m_0)$ and the variance estimator are fixed in subsequent elimination stages. Based on Assumption 1 in Section 3.3 and the similar derivation of Theorem 1 in Tsai and Nelson (2010),

[10] We do not report the results of $\mathcal{AFS}+$ since it performs quite similar to $\mathcal{AFS}$.

Table 3: Performance measures for $\mathcal{AFS}$-$\mathcal{SAA}$ (with nonlinear CV estimation) in comparison with $\mathcal{KN}$ and $\mathcal{TN}$-$\mathcal{SAA}$ in the SC when $m_0 = 20$ or 30, $n_0 = 20$, and $1 - \alpha = 0.95$.

	$R^2_{Y,\mathbf{C}}$	L	$k = 10$		$k = 30$		$k = 50$		$k = 100$	
			PCS	ANS	PCS	ANS	PCS	ANS	PCS	ANS
$\mathcal{KN}$			0.97	316	0.99	401	0.98	438	0.99	508
$\mathcal{TN}$-$\mathcal{SAA}$(20)	0.3		0.96	347	0.95	444	0.97	487	0.95	542
$\mathcal{TN}$-$\mathcal{SAA}$(30)	0.3		0.98	293	0.97	370	0.99	400	0.98	444
$\mathcal{AFS}$-$\mathcal{SAA}$	0.3	20	0.97	182	0.95	219	0.97	235	0.97	258
	0.3	5	0.96	178	0.97	215	0.97	231	0.96	254
	0.3	1	0.97	178	0.97	210	0.98	230	0.98	252
$\mathcal{TN}$-$\mathcal{SAA}$(20)	0.5		0.95	245	0.95	311	0.95	346	0.95	387
$\mathcal{TN}$-$\mathcal{SAA}$(30)	0.5		0.96	216	0.98	271	0.98	292	0.98	329
$\mathcal{AFS}$-$\mathcal{SAA}$	0.5	20	0.95	127	0.97	157	0.97	168	0.97	184
	0.5	5	0.97	125	0.96	155	0.97	165	0.96	179
	0.5	1	0.95	125	0.97	154	0.96	164	0.98	179
$\mathcal{TN}$-$\mathcal{SAA}$(20)	0.7		0.97	157	0.96	199	0.95	221	0.96	243
$\mathcal{TN}$-$\mathcal{SAA}$(30)	0.7		0.98	144	0.98	176	0.97	191	0.98	210
$\mathcal{AFS}$-$\mathcal{SAA}$	0.7	20	0.95	78	0.97	93	0.95	101	0.97	110
	0.7	5	0.95	76	0.97	91	0.97	98	0.97	107
	0.7	1	0.97	73	0.95	90	0.96	96	0.96	106

we can obtain the finite-sample statistical validity of $\mathcal{TN}$-$\mathcal{SAA}$. We choose $m_0 = 20$ and $m_0 = 30$ to investigate the impact of the preliminary-stage sample size on the performance of $\mathcal{TN}$-$\mathcal{SAA}$. Other parameters settings are the same as in the linear CV case. Table 3 summarizes the results of $\mathcal{KN}$, $\mathcal{TN}$-$\mathcal{SAA}$ with $m_0 = 20$ or 30 (denoted by $\mathcal{TN}$-$\mathcal{SAA}(20)$ and $\mathcal{TN}$-$\mathcal{SAA}(30)$, respectively, in the table), as well as $\mathcal{AFS}$-$\mathcal{SAA}$ in the SC.

From Table 3, we notice that $\mathcal{TN}$-$\mathcal{SAA}$, as well as $\mathcal{KN}$ and $\mathcal{AFS}$-$\mathcal{SAA}$, attains the prespecified PCS level in the experiments. However, $\mathcal{TN}$-$\mathcal{SAA}$ may perform worse than $\mathcal{KN}$ if the preliminary-stage sample size m_0 is not appropriately chosen. For instance, when the correlation level is low (i.e., $R^2 = 0.3$), $\mathcal{TN}$-$\mathcal{SAA}$ with $m_0 = 20$ requires a larger sample size than $\mathcal{KN}$. This is probably due to the fact that the SAA estimator $\widehat{\boldsymbol{\beta}}_i(m_0)$ is consistent but biased, and the resulting bias may severely inflate the required sample size especially when m_0 is not large enough. Similar to the findings in the previous sections, the performance of $\mathcal{AFS}$-$\mathcal{SAA}$ is uniformly better than that of $\mathcal{KN}$ and $\mathcal{TN}$-$\mathcal{SAA}$. As expected, there is still little improvement in terms of ANS for $\mathcal{AFS}$-$\mathcal{SAA}$ when increasing the updating frequency of $\widehat{\boldsymbol{\beta}}$.

Table 4: Performance measures for $\mathcal{AFS}$-$\mathcal{SA}$ (with nonlinear CV estimation) in comparison with $\mathcal{KN}$ and $\mathcal{TN}$-$\mathcal{SA}$ in the SC when $m_0 = 30$, $n_0 = 20$, and $1 - \alpha = 0.95$.

	$R^2_{Y,\mathbf{C}}$	L	$k = 10$		$k = 30$		$k = 50$		$k = 100$	
			PCS	ANS	PCS	ANS	PCS	ANS	PCS	ANS
$\mathcal{KN}$			0.97	316	0.99	401	0.98	438	0.99	508
$\mathcal{TN}$-$\mathcal{SA}(30 \times 1)$	0.3		0.98	332	0.98	409	0.98	451	0.99	508
$\mathcal{TN}$-$\mathcal{SA}(5 \times 6)$	0.3		0.97	345	0.97	445	0.96	469	0.96	539
$\mathcal{TN}$-$\mathcal{SA}(2 \times 15)$	0.3		0.95	397	0.98	512	0.97	544	0.98	616
$\mathcal{AFS}$-$\mathcal{SA}$	0.3	20	0.95	207	0.97	248	0.96	267	0.97	283
	0.3	6	0.98	202	0.97	242	0.96	252	0.99	276
	0.3	2	0.97	211	0.97	250	0.98	265	0.97	289
$\mathcal{TN}$-$\mathcal{SA}(30 \times 1)$	0.5		0.98	298	0.97	391	0.99	432	0.97	467
$\mathcal{TN}$-$\mathcal{SA}(5 \times 6)$	0.5		0.97	320	0.95	428	0.96	443	0.96	496
$\mathcal{TN}$-$\mathcal{SA}(2 \times 15)$	0.5		0.96	363	0.95	473	0.95	525	0.95	584
$\mathcal{AFS}$-$\mathcal{SA}$	0.5	20	0.96	172	0.96	196	0.97	207	0.97	225
	0.5	6	0.98	168	0.98	192	0.96	201	0.96	219
	0.5	2	0.97	177	0.98	205	0.98	216	0.99	233
$\mathcal{TN}$-$\mathcal{SA}(30 \times 1)$	0.7		0.96	309	0.95	386	0.95	394	0.96	443
$\mathcal{TN}$-$\mathcal{SA}(5 \times 6)$	0.7		0.96	336	0.95	381	0.95	403	0.96	458
$\mathcal{TN}$-$\mathcal{SA}(2 \times 15)$	0.7		0.96	364	0.95	448	0.95	475	0.95	530
$\mathcal{AFS}$-$\mathcal{SA}$	0.7	20	0.97	137	0.98	156	0.96	168	0.98	177
	0.7	6	0.98	133	0.96	152	0.98	166	0.98	175
	0.7	2	0.98	145	0.97	160	0.98	170	0.98	182

For $\mathcal{TN}$-$\mathcal{SA}$, we compute the estimator $\widehat{\boldsymbol{\beta}}_i(m_0)$ as in the SA algorithm (i.e., Algorithm 2) described in Section 2.2, based on a set of preliminary-stage m_0 samples. That is, in the preliminary stage, the SA algorithm implements n iterations, each consists of m observations, to recursively update $\widehat{\boldsymbol{\beta}}_i$ (i.e., $m \times n = m_0 = 30$). We specify different settings of $m \times n$ in order to evaluate their impact on the performance of $\mathcal{TN}$-$\mathcal{SA}$. The values of coefficient estimator $\widehat{\boldsymbol{\beta}}_i(m_0)$ and variance estimator of $\mathcal{TN}$-$\mathcal{SA}$ are both fixed in subsequent elimination stages. Other parameter settings remain the same. Table 4 summarizes the results of $\mathcal{KN}$, $\mathcal{TN}$-$\mathcal{SA}$ with different $m \times n$ (denoted by $\mathcal{TN}$-$\mathcal{SA}(30 \times 1)$, $\mathcal{TN}$-$\mathcal{SA}(5 \times 6)$, and $\mathcal{TN}$-$\mathcal{SA}(2 \times 15)$, respectively, in the table), as well as $\mathcal{AFS}$-$\mathcal{SA}$ in the SC.

From Table 4, we have several interesting findings. First, we notice that a larger value of m makes $\mathcal{TN}$-$\mathcal{SA}$ more efficient in terms of reduced ANS, which could be due to the fact that the variance of the gradient estimator $g(\widehat{\boldsymbol{\beta}})$ decreases as m increases, thus leading to smaller variance of $\widehat{\theta}_{\mathrm{SA}}$. Second, it is also interesting to point out that the value of ANS in $\mathcal{AFS}$-$\mathcal{SA}$ initially decreases and then increases when the value of L is decreased, which implies that choosing an appropriate value of L becomes a sophisticated task for $\mathcal{AFS}$-$\mathcal{SA}$. Third, even though in this situation, $\mathcal{AFS}$-$\mathcal{SA}$ in general performs much better than both $\mathcal{KN}$ and $\mathcal{TN}$-$\mathcal{SA}$. Last but not least, when comparing Table 3 with Table 4, we find that the performance of $\mathcal{AFS}$-$\mathcal{SAA}$ tends to be superior to $\mathcal{AFS}$-$\mathcal{SA}$ in terms of ANS, and both procedures achieve the requested PCS level. We further evaluate the experimental performance of $\mathcal{AFS}$-$\mathcal{SA}$ with different choices of the step-size sequence $a_w = z/w$, where $z = \{0.1, 0.2, \ldots, 2.0\}$, in Table 5. We find that $\mathcal{AFS}$-$\mathcal{SA}$ achieves the PCS nominal value of 0.95 in all settings, and reveals the best performance in terms of ANS when $z = 0.5$ or 0.8. The ANS performance is getting worse when the value of z becomes larger, but is still better than both $\mathcal{KN}$ and $\mathcal{TN}$-$\mathcal{SA}$.

5 Illustrative Examples

In this section, we consider two illustrative examples, i.e., the mean time to failure (MTTF) problem in a reliability system and the equity investment problem in a financial engineering system, which can be modeled as a continuous-time Markov chain (CTMC) with finite state space and a discrete-time Markov chain (DTMC) with infinite state space, respectively. We examine the efficiency of our proposed adaptive procedures in comparison with conventional $\mathcal{KN}$ and $\mathcal{TN}$-type procedures.

5.1 The MTTF Problem

For simplicity, we suppose that the reliability system consists of two components that work as an active and a spare component. The spare component becomes active when the current active

Table 5: Performance measures for $\mathcal{AFS}$-$\mathcal{SA}$ (with nonlinear CV estimation) in the SC with different settings of the step-size sequence $a_w = z/w$, where $z = \{0.1, 0.2, \ldots, 2.0\}$, when $R^2_{Y,\mathbf{C}} = 0.3$, $L = 20$, $k = 30$, $n_0 = 20$, and $1 - \alpha = 0.95$.

z	PCS	ANS	z	PCS	ANS
0.1	0.96	315	1.1	0.98	258
0.2	0.98	279	1.2	0.97	266
0.3	0.97	260	1.3	0.97	271
0.4	0.96	251	1.4	0.97	281
0.5	0.98	247	1.5	0.97	285
0.6	0.97	250	1.6	0.97	295
0.7	0.96	249	1.7	0.98	303
0.8	0.98	247	1.8	0.97	314
0.9	0.95	250	1.9	0.97	317
1.0	0.97	262	2.0	0.97	327

component fails, while the failed component immediately commences repair. The failed component becomes the spare when its repair is completed. Only one component at a time can be repaired, therefore the system fails as long as both components have failed (see Nelson (2013) for more detailed introduction).

For tractability, the failure times and the repair times of the components are assumed to be independently and exponentially distributed so that the system may be modeled as a CTMC. Let $Z = \{Z(t) : t \geq 0\}$ denote the number of functional components at time t, defined on the finite state space $\Sigma = \{0, 1, 2\}$. Suppose that Z reaches the absorbing state 0 almost surely starting from $Z(0) = 2$ and let $T = \inf\{t \geq 0 : Z(t) = 0 | Z(0) = 2\}$ be the time till absorption (i.e., time to failure). We consider $k = 10$ systems, corresponding to 10 configurations of the failure rate (denote by λ_i) and repair rate (denote by γ_i), $i = 1, 2, \ldots, k$. Accordingly, let Z_{ij} and T_{ij} denote the sample path of system states and the system failure time from the jth replication of system i. Then, the MTTF (i.e., $\mathrm{E}[T_{ij}]$) can be computed analytically, as presented in descending order in Table 6. Notice that a system with a larger MTTF is better, so that system 1 is the best system. We want to select system 1 with the proposed and existing R&S procedures, where the parameters λ_i and γ_i are known for each configuration (assuming that their MTTF are unknown and need to be estimated by simulation). The choices of effective linear and nonlinear control variates are introduced in the following subsection.

5.1.1 Constructions of Linear and Nonlinear Control Variates

When using the linear CV estimation, we simply let the simulation output $Y_{ij} = T_{ij}$, and consider two possible choices of the controls. The first used control $C^{(1)}_{ij}$ is the average lifetime of a component

Table 6: The ten configurations of Markovian reliability systems and their MTTF.

system i	1	2	3	4	5	6	7	8	9	10
λ_i	1.00	1.04	0.96	0.99	1.03	1.02	0.98	0.95	1.01	0.97
γ_i	8.0	8.4	7.0	7.5	8.2	8.0	7.3	6.8	7.8	7.1
MTTF	10.000	9.689	9.679	9.672	9.671	9.650	9.642	9.640	9.627	9.608

during the sample path Z_{ij} (with $\mathrm{E}[C_{ij}^{(1)}] = 1/\lambda_i$). For instance, suppose that there are 50 component failures of the sample path Z_{ij}, and their life durations can be denoted as $\{D_1, D_2, \ldots, D_{50}\}$. Then, we can compute $C_{ij}^{(1)} = 1/50 \sum_{d=1}^{50} D_d$. The other used control, denote by $C_{ij}^{(2)}$, is the number of times the sample path Z_{ij} enters state 1, which follows a geometric distribution with parameter $\lambda_i/(\lambda_i + \gamma_i)$. Then, we have $\mathrm{E}[C_{ij}^{(2)}] = (\lambda_i + \gamma_i)/\lambda_i$.

When using the nonlinear CV estimation, we need to introduce the approximating martingales technique in Henderson and Glynn (2002) in order to construct a nonlinearly parameterized family of control variates. In the following, we illustrate the construction step-by-step, aiming to provide some guidelines as applying to more complicated MTTF problems or even other types of CTMC problems.

As mentioned by Ahamed et al. (2006), the MTTF problem can be regarded as optimizing the *expected cost prior to absorption* for finite state-space Markov chains. That is, we let $c : \Sigma \to \mathbb{R}$ be a given cost function and then define the expected cost accrued until absorption as $\theta(x) = \mathrm{E}\big[\int_0^T c\,(Z(s))\,\mathrm{d}s | Z(0) = x\big]$, for all $x \in \Sigma \backslash \{0\}$, and set $\theta(0) = 0$. Then, we have $\theta(x) = \mathrm{MTTF}$ when setting $c(Z) = 1$ and $x = 2$. We present the following proposition of Henderson and Glynn (2002), which is applied to estimate the expected cost prior to absorption, for identifying an appropriate class of approximating martingales, and based on which the nonlinear function $f\big(\mathbf{C}_{ij}, \widehat{\boldsymbol{\beta}}_i\big)$ can be constructed accordingly.

Proposition 1. *(Proposition 2 in Henderson and Glynn (2002)) Let $u : \Sigma \to \mathbb{R}$ be a real-valued function on the state space Σ with $u(0) = 0$, and for $t \geq 0$ let*

$$M(t) = u\,(Z(t)) - u\,(Z(0)) - \int_0^t Gu\,(Z(s))\,\mathrm{d}s, \tag{5}$$

where G is the transition rate matrix of the CTMC $\{Z(t), t \geq 0\}$. Then, for any function $u(\cdot)$, the stochastic process $M(\cdot)$ is a Dynkin martingale. Moreover, if $E[T|Z(0) = x] < \infty$, then

$$E[M(T)|Z(0) = x] = 0.$$

Suppose that the basis function $u(Z) = u(Z; \boldsymbol{\beta}) = \beta_1 Z^{\beta_2}$, with the dimension of $\boldsymbol{\beta}$ being $p = 2$,

which has been successfully used and recommended by Kim and Henderson (2004). Then, the last term in Equation (5) is an integral function based on G and $u(\cdot)$. Interested readers may refer to Henderson and Glynn (2002) for more details of computation. It should be noticed that Proposition 1 claims that the expectation of $M(T)$ is zero, which means it can be served as a parameterized control variate function. Therefore, we let $\widehat{\boldsymbol{\beta}}_i = (\widehat{\beta}_{i1}, \widehat{\beta}_{i2})^{\mathtt{T}}$ and the nonlinear function $f(\mathbf{C}_{ij}, \widehat{\boldsymbol{\beta}}_i)$ can be formulated as follows,

$$\begin{aligned} f(\mathbf{C}_{ij}, \widehat{\boldsymbol{\beta}}_i) &= M(T_{ij}) \\ &= u(Z_{ij}(T_{ij})) - u(Z_{ij}(0)) - \int_0^{T_{ij}} Gu(Z_{ij}(s))\, \mathrm{d}s \\ &= 0 - \widehat{\beta}_{i1} 2^{\widehat{\beta}_{i2}} - T_{ij}^{(1)} \left(\gamma_i \widehat{\beta}_{i1} 2^{\widehat{\beta}_{i2}} - (\lambda_i + \gamma_i) \widehat{\beta}_{i1} \right) - T_{ij}^{(2)} \left(\lambda_i \widehat{\beta}_{i1} - \lambda_i \widehat{\beta}_{i1} 2^{\widehat{\beta}_{i2}} \right). \end{aligned} \tag{6}$$

Equation (6) holds because $Z_{ij}(T_{ij}) = 0$, $Z_{ij}(0) = 2$, and the last two terms follows from the calculation of the integral function, where $T_{ij}^{(1)}$ and $T_{ij}^{(2)}$ represent the total time $Z_{ij}(t)$ stays in state 1 and 2, respectively.

Given the constructions of linear and nonlinear CVs, we are ready to set the parameters in the MTTF problem.

5.1.2 Problem Settings and Simulation Results

For $\mathcal{KN}$, $\mathcal{TN}$-type, and the proposed adaptive procedures, we all set the first-stage sample size $n_0 = 20$. In addition, we set the preliminary-stage sample size $m_0 = 30$ for all $\mathcal{TN}$-type procedures, and the $\widehat{\boldsymbol{\beta}}$ updating frequency $L = 20$ for our proposed procedures.

When using the linear CV estimation, we let $\mathcal{TN}$-1 and $\mathcal{TN}$-2 denote the $\mathcal{TN}$ procedures being applied with the controls $C_{ij}^{(1)}$ and $C_{ij}^{(2)}$, respectively. Our proposed procedures with linear CV approach, i.e., $\mathcal{AFS}$ and $\mathcal{AFS}$+, will be applied with the control $C_{ij}^{(2)}$ (we will explain the reason later). When using the SA algorithm in either $\mathcal{TN}$-$\mathcal{SA}$ or $\mathcal{AFS}$-$\mathcal{SA}$, we set the initial solution $\widehat{\boldsymbol{\beta}}_{i0} = \boldsymbol{-1}$ and the step-size sequence $a_w = 0.05/w$. In $\mathcal{TN}$-$\mathcal{SA}$, we implement $n = 6$ iterations with each consists of $m = 5$ observations, to compute $\widehat{\boldsymbol{\beta}}_i(m_0)$ in the preliminary stage.

The nominal PCS is set to $1 - \alpha = 0.95$. The IZ parameter is set to $\delta = 0.311$, which equals to the difference between the MTTFs of systems 1 and 2, and thus the correct selection is to choose system 1. In addition to the performance measures of PCS and ANS, we also report the CPU running time per replication (in seconds), based on 1,000 complete macro-replications for all procedures, in Table 7.

In the following, we explain the interesting findings from Table 7. First of all, the observed PCS for all procedures exceeds the nominal value of 0.95. To check the normality of T_{ij} and

Table 7: Results for the proposed procedures applied to the MTTF problem in comparison with $\mathcal{KN}$ and $\mathcal{TN}$-type procedures in 1,000 trials with $\delta = 0.311$ and $1 - \alpha = 0.95$.

Procedure	PCS	ANS	CPU time (sec.)	Procedure	PCS	ANS	CPU time (sec.)
$\mathcal{KN}$	0.964	6,863	28.21	$\mathcal{TN}$-$\mathcal{SAA}$	0.985	3,088	13.44
$\mathcal{TN}$-1	0.963	6,108	26.06	$\mathcal{TN}$-$\mathcal{SA}$	0.964	4,941	15.59
$\mathcal{TN}$-2	0.984	3,637	13.64	$\mathcal{AFS}$-$\mathcal{SAA}$	0.982	1,784	9.12
$\mathcal{AFS}$	0.984	2,645	11.19	$\mathcal{AFS}$-$\mathcal{SA}$	0.976	3,402	11.91
$\mathcal{AFS}$+	0.982	2,572	10.96				

$C_{ij}^{(1)}$, we implement Shapiro-Wilk goodness of fit test (cf., Royston (1993)) using MATLAB with a significance level of 0.05. The results suggest that both T_{ij} and $C_{ij}^{(1)}$ are intrinsically not normal, but can be well approximated by normal distribution when taking batches with 50 samples per batch. This illustrates the robustness of our procedures when the normality assumption is violated. Second, in terms of ANS, $\mathcal{TN}$-1 and $\mathcal{TN}$-2 are both superior to $\mathcal{KN}$, meanwhile $\mathcal{TN}$-2 is much superior to $\mathcal{TN}$-1. The difference in the efficiency between $\mathcal{TN}$-1 and $\mathcal{TN}$-2 can be explained by the estimated correlations between Y_{ij} and $C_{ij}^{(1)}$ or $C_{ij}^{(2)}$ based on $10,000$ replications (for system 1), which is $R^2_{Y,C^{(1)}} = 0.19$ and $R^2_{Y,C^{(2)}} = 0.63$, respectively. It is also the reason why our adaptive fully sequential procedures with linear CV estimation, i.e., $\mathcal{AFS}$ and $\mathcal{AFS}$+, adopt $C_{ij}^{(2)}$ as the controls. Third, in fact, both $\mathcal{AFS}$ and $\mathcal{AFS}$+ can further yield a significant improvement in terms of ANS over the ordinary fully sequential procedure. For instance, there is a reduction of 27% and 29% in the ANS of $\mathcal{AFS}$ and $\mathcal{AFS}$+ even when compared to $\mathcal{TN}$-2. Fourth, however, the reduction in CPU time compared to $\mathcal{TN}$-2 is only 18% and 19% for $\mathcal{AFS}$ and $\mathcal{AFS}$+, respectively, which is due to the fact that the adaptive procedures implement more iterations of linear regressions. It should also be noticed that, in this example, the computational overhead of $\mathcal{AFS}$+ for computing the point and variance estimators, whenever $\widehat{\boldsymbol{\beta}}$ is updated, is not as significant as expected, because we have only 10 competing systems.

Moreover, similar to the numerical results in the previous section, the SAA approach demonstrates better performance than the SA approach when being employed in adaptive procedures, which is also consistent with the comparative results for the applications regarding nonlinear CV estimation in Kim and Henderson (2007). In Table 7, we can observe that there is a significant reduction of 42% in terms of ANS from $\mathcal{TN}$-$\mathcal{SAA}$ to $\mathcal{AFS}$-$\mathcal{SAA}$. If we compare the performance of all procedures, we can see that $\mathcal{AFS}$-$\mathcal{SAA}$ requires the fewest observations and therefore the smallest CPU time. On one hand, the adaptive procedures with linear CV estimation is easily-implemented, but its performance may depend heavily on an appropriate choice of effective controls. On the other hand, the application of nonlinear CV estimation is not so straightforward, but it

may achieve a more substantial saving in sampling cost.

Talking about the sampling cost, it should be noted that the CPU running time is mainly spent on two tasks: (i) generating a number of simulated observations from each system (which we denote as the sampling cost), and (ii) executing specific method to compute $\widehat{\boldsymbol{\beta}}_i$ and then obtain $\bar{\theta}_i(r)$ (which we denote as the computing cost). We also calculate the computing costs of performing task (ii) in $\mathcal{AFS}$, $\mathcal{AFS}$-$\mathcal{SAA}$ and $\mathcal{AFS}$-$\mathcal{SA}$, which involves implementing the linear regression for $\mathcal{AFS}$, the nonlinear optimization solver for $\mathcal{AFS}$-$\mathcal{SAA}$, and the SA recursive algorithm for $\mathcal{AFS}$-$\mathcal{SA}$. The corresponding computational time values are 1.89, 4.75 and 2.75 seconds, respectively. We observe that the computing cost accounts for 17%, 52%, and 23% of the overall CPU time for these three adaptive procedures. The nonlinear optimization used in the SAA algorithm can be computationally intensive relative to the linear regression or the SA iterative algorithm, but it may occupy a small fraction of the required effort for long simulation runs. In fact, in many large-scale or complicated simulation problems (for which VRTs are most needed), the computing cost should be small relative to the sampling cost because the simulated observations themselves are expensive, as argued in Avramidis and Wilson (1996). For such problems, the execution of optimization solver or algorithm rarely dominates the overall running time.

5.2 The Equity Investment Problem

In this subsection, we consider an equity investment problem. Suppose that a fund manager has a certain amount of money and wants to invest all of them in one company. Each company has an initial asset price and this price changes with time. Specifically, we assume that the asset price obeys a stochastic process model. The manager can only make the profit (say a dividend) when the asset price of the company exceeds a preset threshold K at a preset date D, and the amount of the profit is $\mathrm{E}[(A(D)-K)^+]$, where $A(D)$ is the asset price at date D and $x^+ = \max\{0, x\}$. While there are several candidate companies, the fund manager wants to invest the company with the highest expected profit return, i.e., the one with the largest $\mathrm{E}[(A(D)-K)^+]$.

The following problem description is primarily adapted from Kim and Henderson (2007). Let $\{A(t), t \geq 0\}$ denote the asset price of the company at time t, which is assumed to follow a geometric Brownian motion process with constant parameters c_r and c_v (known as the risk-free interest rate and volatility, respectively). Suppose that the asset price is monitored at a discrete time sequence $\{t_d = d\Delta t, d = 0, 1, \ldots, T\}$, where $\Delta t = D/T$ is the time between consecutive monitoring dates. For notational simplification, we let A_d represent the underlying asset price at the dth monitoring point (i.e., $A_d = A(t_d)$). Similar to the barrier option setting in Kim and Henderson (2007), we also assume that there is an upper barrier level H_u and a lower barrier level H_ℓ, where $H_\ell < K < H_u$. If the asset price A_d reaches the barriers, i.e., $A_d \notin [H_\ell, H_u]$, then the profit is zero and the manager

Table 8: The ten configurations of company's assets and their expected profit returns.

system i	1	2	3	4	5	6	7	8	9	10
A_0	130	125	120	115	110	105	100	95	90	85
K	125	120	115	110	105	100	95	90	85	80
$U_T^*(A_0, K)$	6.81	9.08	10.57	11.27	11.40	11.23	10.93	10.60	10.27	9.93

gets nothing. If the asset price has not crossed $[H_\ell, H_u]$ by the terminal time D, then the profit the manager gets is $(A_T - K)^+$. Suppose that the initial asset price A_0 is within the interval $[H_\ell, H_u]$. Then, the price A_d at time d, $d = 1, 2, \ldots, T$, can be expressed in the following recursive form,

$$A_d = A_{d-1}\exp\left\{(c_r - c_v^2/2)\Delta t + c_v\sqrt{\Delta t}B_d\right\}, \quad d = 1, 2, \ldots, T,$$

where $B_1, \ldots, B_T$ are i.i.d. standard normal random variables $N(0, 1)$. Note that the profit that the manager can get depends on the complete path $\{A_d, d = 0, 1, \ldots, T\}$, and the expected profit is given by

$$\exp(-c_r D)\mathrm{E}[\mathbb{1}_{\{\tau > T\}}(A_T - K)^+]$$

where $\tau = \inf\{d \geq 0 : A_d \notin [H_\ell, H_u]\}$ and $\mathbb{1}_{\{E\}} = 1$ if the event E is true, otherwise it equals to 0. Since the discount factor $e^{-c_r D}$ is constant, the problem reduces to estimating the expected profit with the initial asset price A_0. Namely, our goal is to estimate $U_T^*(A_0, K)$, which is defined as

$$U_T^*(A_0, K) = \begin{cases} \mathrm{E}\left[\mathbb{1}_{\{\tau > T\}}(A_T - K)^+ | A_0\right], & \text{if } A_0 \in [H_\ell, H_u], \\ 0, & \text{if } A_0 = 0 \text{ or } A_0 \notin [H_\ell, H_u]. \end{cases}$$

Following the experimental settings of Kim and Henderson (2007), we set the parameters $D = 1$, $T = 6$, $c_r = 0.05$, $c_v = 0.1$, and $[H_\ell, H_u] = [50, 150]$. We consider $k = 10$ companies, corresponding to 10 systems with different configurations of the initial asset price A_0 and the preset threshold K along with their expected profit (i.e., $U_T^*(A_0, K)$), which are estimated with 10^7 replications, as presented in Table 8. Notice that a system with a larger expected profit is better, so that system 5 is the best system.

In order to demonstrate how to apply $\mathcal{AFS}$-$\mathcal{SAA}$ and $\mathcal{AFS}$-$\mathcal{SA}$ to solve the investment problem, we need to introduce another approximating martingales technique in Henderson and Glynn (2002) to construct the nonlinear control variates. Let $\widetilde{A}_d = \mathbb{1}_{\{\tau > d\}}A_d$, for $d = 1, 2, \ldots, T$. Then $\{\widetilde{A}_d : 0 \leq d \leq T\}$ is a DTMC on the state space $\Sigma = [H_\ell, H_u] \cup \{0\}$ (assuming that $A_0 \in \Sigma$). We present Proposition 2, based on which the nonlinear function $f\big(\mathbf{C}_{ij}, \widehat{\boldsymbol{\beta}}_i\big)$ can be constructed accordingly.

Proposition 2. *(Proposition 5 in Henderson and Glynn (2002)) Let $u_d : \Sigma \to \mathbb{R}$ be a sequence of*

bounded real-valued functions on the state space Σ *with* $u_d(0) = 0$*, where* $d = 0, 1, \ldots, T-1$*. Let* $M_0 = 0$*, and for* $1 \le t \le T$ *let*

$$M_t = \sum_{d=1}^{t} \left[u_{T-d}(\widetilde{A}_d) - P(\widetilde{A}_{d-1}, \cdot) u_{T-d}(\cdot) \right], \tag{7}$$

where P *is the transition probability kernel of the DTMC* $\{\widetilde{A}_d : 0 \le d \le T-1\}$*, which is defined as* $P(x, \cdot)c(\cdot) = \mathrm{E}[c(\widetilde{A}_1)|\widetilde{A}_0 = x]$ *for any real-valued function* $c : \Sigma \to \mathbb{R}$*. Then, for any sequence of functions* $\{u_d : 0 \le d \le T\}$*, the stochastic process* $\{M_t : 0 \le t \le T\}$ *is a martingale.*

As suggested by Kim and Henderson (2007)[11], a simple parameterization for the basis functions $u_s(\cdot) = u_s(\cdot; \widehat{\boldsymbol{\beta}})$, $s = 0, 1, \ldots, T-1$, is given by

$$u_s(x; \widehat{\boldsymbol{\beta}}) = \begin{cases} 0, & \text{if } x = 0, \\ (x - K)^+, & \text{if } s = 0, \\ x^{\widehat{\beta}_{4(s-1)+2}} \widehat{\beta}_{4(s-1)+1} + x \widehat{\beta}_{4(s-1)+3} + \widehat{\beta}_{4s}, & \text{if } s = 1, 2, \ldots, T-1 \text{ and } x \neq 0, \end{cases} \tag{8}$$

where $x = \widetilde{A}_d$, $s = T - d$, and $\widehat{\boldsymbol{\beta}} = (\widehat{\beta}_1, \widehat{\beta}_2, \ldots, \widehat{\beta}_{4(T-1)})^{\mathtt{T}}$. It can be shown that $\mathrm{E}[M_T(\widehat{\boldsymbol{\beta}})] = 0$ for any $\widehat{\boldsymbol{\beta}}$, where M_T is given by Equation (7) with u_s specified in Equation (8) (cf., Henderson and Glynn (2002)). Then, we set $f(\mathbf{C}_{ij}, \widehat{\boldsymbol{\beta}}_i) = M_T$, which is the nonlinear function for system i in replication j. Therefore, $(\widetilde{A}_T - K)^+ - M_T$ can be regarded as a nonlinear CV estimator of $U_T^*(A_0, K)$.

We compare our proposed procedures, $\mathcal{AFS}$-$\mathcal{SAA}$ and $\mathcal{AFS}$-$\mathcal{SA}$, with $\mathcal{KN}$, in which the parameters are set as follows. We set the first-stage sample size $n_0 = 20$ for the three procedures, and set the $\widehat{\boldsymbol{\beta}}$ updating frequency $L = 20$ and initial solution $\widehat{\boldsymbol{\beta}}_{i0} = \mathbf{0}$ for both $\mathcal{AFS}$-$\mathcal{SAA}$ and $\mathcal{AFS}$-$\mathcal{SA}$. In addition, for $\mathcal{AFS}$-$\mathcal{SA}$, we need to specify the step-size sequence as $a_w = \frac{5(10)^{-4}}{10^5 + w^{2/3}}$. The nominal PCS is set to $1 - \alpha = 0.95$. The IZ parameter is set to $\delta = 0.13$, which equals to the difference between the expected profit return from systems 4 and 5, and thus the correct selection is to choose system 5. We report the performance measures of PCS, ANS, and CPU running time per replication (in seconds), based on 1,000 complete macro-replications for all procedures, in Table 9. Similar to the previous numerical results, the observed PCS for all procedures exceeds the nominal value of 0.95. Both $\mathcal{AFS}$-$\mathcal{SAA}$ and $\mathcal{AFS}$-$\mathcal{SA}$ require much smaller number of observations and shorter CPU running time than $\mathcal{KN}$, and $\mathcal{AFS}$-$\mathcal{SAA}$ still performs better than $\mathcal{AFS}$-$\mathcal{SA}$.

[11] Interested readers may refer to the Appendix of Kim and Henderson (2007) for the detailed parameterizations.

Table 9: Results for the proposed procedures applied to the equity investment problem in comparison with $\mathcal{KN}$ in 1,000 trials with $\delta = 0.13$ and $1 - \alpha = 0.95$.

Procedure	PCS	ANS	CPU (sec.)
$\mathcal{KN}$	0.981	3,779	39.72
$\mathcal{AFS}$-$\mathcal{SAA}$	0.979	484	21.58
$\mathcal{AFS}$-$\mathcal{SA}$	0.991	936	26.24

6 Conclusions

In this paper, we develop adaptive fully sequential selection procedures with linear or nonlinear CV estimation. In the proposed procedures, the control coefficient estimator $\widehat{\boldsymbol{\beta}}$ for each surviving system is updated sequentially as the sampling process progresses. We prove the statistical validity of adaptive procedures with linear or nonlinear CV estimation in the asymptotic regime. We also provide proofs for the SLLN and CLT of the SAA nonlinear CV estimator used in the adaptive procedures, which is nontrivial and different from the existing literature. The experimental studies reveal that the proposed procedures are superior to the ordinary fully sequential procedures (based on sample means or CS estimators as in $\mathcal{KN}$ and $\mathcal{TN}$ procedures) in terms of reduced sampling cost to achieve the required PCS level. The empirical results also indicate that the application of SAA-based estimator delivers a more robust performance (compared to SA-based estimator) when using the nonlinear CV estimation. The finite-time performance of the SA-based estimator is strongly impacted by certain tuning parameters (e.g., the initial solution and step-size parameters) and the selections of which are nontrivial. Overall, we find that it is unnecessary to very frequently update $\widehat{\boldsymbol{\beta}}$ in the adaptive procedures, and thus the computational overhead should not be increased much beyond that of the non-adaptive CV-based procedures by Tsai and Nelson (2010).

There are several possible directions of future research. For instance, we can employ the adaptive linear or nonlinear CV scheme in R&S procedures for steady-state simulation with a single replication (cf., Kim and Nelson (2006a)). Another possible extension is to consider combined schemes of adaptive CV and importance sampling techniques (cf., Ahamed et al. (2006)) for use in R&S procedures when estimating performance measures related to rare events. It is also worthwhile considering to apply the proposed adaptive procedures to solve large-scale R&S problems for practical complicated Markovian systems, e.g., the treatment selection of the simulation-based personal medicine care, and the network design of integrated supply chain management.

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Appendix

A Proofs of Theorems 1 and 2

In this section, we prove Theorems 1 and 2 which show the statistical validity of $\mathcal{AFS}$ procedure as implementing the $\widehat{\boldsymbol{\beta}}$ updating Algorithms 3 and 4. In order to achieve that, we first need to properly scale the discrete-time process $\bar{\theta}_i(r) - \bar{\theta}_h(r)$, which is used in the elimination step. Then, the triangular region, constructed also for elimination, will also be scaled in the same limiting regime. Based on that, we can prove the desired PCS guarantee. The proofing framework is quite similar to the works of Luo et al. (2015) and Tsai et al. (2017), however, the detailed constructions are substantially different.

Recall that, in $\mathcal{AFS}$ procedure, alternative i is eliminated if there exists some other surviving alternative h such that

$$\bar{\theta}_i(r) - \bar{\theta}_h(r) < \min\left\{0, -\frac{a}{\delta} \cdot \frac{\left[S_i^2(r) + S_h^2(r)\right]}{r} + \frac{\delta}{2}\right\}, \tag{9}$$

where $\bar{\theta}_i(r)$ and $S_i^2(r)$ are the sample mean and sample variance of alternative i defined in Equation (4), in which each observation $\widehat{\theta}_{ij}$ is calculated as $\widehat{\theta}_{ij} = Y_{ij} - (\mathbf{C}_{ij} - \boldsymbol{\mu}_i)^{\mathrm{T}} \widehat{\boldsymbol{\beta}}_{i,\ell-1}$, where $\widehat{\boldsymbol{\beta}}_{i,\ell-1}$ is updated as $\widehat{\boldsymbol{\beta}}_{i,\ell-1} = \mathbf{S}_{\mathbf{C}_i}^{-1}(r)\mathbf{S}_{\mathbf{C}_i Y_i}(r)$ in the linear CV approach, and $\widehat{\theta}_{ij} = Y_{ij} - f\big(\mathbf{C}_{ij}, \widehat{\boldsymbol{\beta}}_{i,\ell-1}\big)$, where $\widehat{\boldsymbol{\beta}}_{i,\ell-1}$ is one of the first-order critical points by solving $\arg\min_{\boldsymbol{\beta}_i} \mathrm{Var}(\boldsymbol{\beta}_i, r) = \arg\min_{\boldsymbol{\beta}_i} \frac{1}{r-1}\left[\sum_{j=1}^{r}\big(\widehat{\theta}_{ij}\big)^2 - \frac{1}{r}\Big(\sum_{j=1}^{r}\widehat{\theta}_{ij}\Big)^2\right]$ in the SAA approach. Since the linear CV is a special case of nonlinear SAA approach, we would like to adopt the equation $\widehat{\theta}_{ij} = Y_{ij} - f\big(\mathbf{C}_{ij}, \widehat{\boldsymbol{\beta}}_{i,\ell-1}\big)$ when they can be unified, but present the results for them separately when the proofing techniques are quite different.

For notational simplicity, we introduce the $\widehat{\boldsymbol{\beta}}$ updating sequence $\{r_\ell : r_0 = 0, r_1 = n_0 < r_2 < r_3 < \cdots\}_{\ell=0}^{\infty}$ to be the counter points at which $\widehat{\boldsymbol{\beta}}_{i\ell}$ will be updated. It is easy to see that $r_\ell - r_{\ell-1} = L$ for all $\ell \geq 2$ as in $\mathcal{AFS}$ procedure.[12] Then, the jth observation from system i, when $j \in (r_{\ell-1}, r_\ell]$ for some $\ell \geq 1$, is

$$\widehat{\theta}_{ij} = Y_{ij} - f\big(\mathbf{C}_{ij}, \widehat{\boldsymbol{\beta}}_{i,\ell-1}\big) \tag{10}$$

Therefore, at stage t, $t = 1, 2\ldots$, with r total samples from system i, i.e., $r \in (r_{t-1}, r_t]$, we know that $\widehat{\boldsymbol{\beta}}_i$ has been updated for $(t-1)$ times and the sample-mean estimator $\bar{\theta}_i(r) = \sum_{j=1}^{r}\widehat{\theta}_{ij}$ is always unbiased with mean $\mathrm{E}[\bar{\theta}_i(r)] = \theta_i$ under certain conditions. At the first stage $t = 1$, $\widehat{\theta}_{ij} = Y_{ij}$ due to that $f\big(\mathbf{C}_{ij}, \widehat{\boldsymbol{\beta}}_{i0}\big) = 0$, which are i.i.d. random variables with mean θ_i and variance σ_i^2. At other stages $t \geq 2$, we summarize the result in the following lemma.

[12] Note that the $\widehat{\boldsymbol{\beta}}$ updating sequence could be more flexible as long as it satisfies the necessary conditions in the asymptotic regime.

Lemma 1 (SLLN for $\bar{\theta}_i(r)$)**.** *Suppose that all assumptions in Theorems 1 and 2 are satisfied. In particular, we assume the initial value of* $\widehat{\boldsymbol{\beta}}_{i0}$ *such that* $f\big(\mathbf{C}_{ij}, \widehat{\boldsymbol{\beta}}_{i0}\big) = 0$ *for all system* $i = 1, 2, \ldots, k$ *and the first-stage sample size* $n_0 \to \infty$ *and* $\frac{n_0}{r} \to 0$ *as the sample size* $r \to \infty$ *in* $\mathcal{AFS}$ *procedure. Suppose that* $\widehat{\boldsymbol{\beta}}_{i1} \xrightarrow{a.s.} \boldsymbol{\beta}_i^*$*, for some fixed* $\boldsymbol{\beta}_i^* \in \mathbf{B}$*. Then, as* $r \to \infty$*,* $\bar{\theta}_i(r) \xrightarrow{a.s.} \theta_i$*.*

Note that the condition that $\widehat{\boldsymbol{\beta}}_{i1} \xrightarrow{\text{a.s.}} \boldsymbol{\beta}_i^*$, for some fixed $\boldsymbol{\beta}_i^* \in \mathbf{B}$, can be satisfied under Assumption 4. Following the main steps in proving Theorem 5.2 in Kim and Henderson (2007), we can prove the result in a similar way.

Proof. At any stage $t \geq 2$ with sample size $r > n_0$, the sample mean estimator

$$\begin{aligned}
\bar{\theta}_i(r) &= \frac{1}{r}\left[\sum_{j=1}^{n_0} \widehat{\theta}_{ij} + \sum_{\ell=2}^{t-1} \sum_{j=r_{\ell-1}+1}^{r_\ell} \widehat{\theta}_{ij} + \sum_{j=r_{t-1}+1}^{r} \widehat{\theta}_{ij}\right] \\
&= \frac{1}{r}\left[\sum_{j=1}^{n_0} Y_{ij} + \sum_{\ell=2}^{t-1} \sum_{j=r_{\ell-1}+1}^{r_\ell} \Big(Y_{ij} - f\big(\mathbf{C}_{ij}, \widehat{\boldsymbol{\beta}}_{i,\ell-1}\big)\Big) + \sum_{j=r_{t-1}+1}^{r} \Big(Y_{ij} - f\big(\mathbf{C}_{ij}, \widehat{\boldsymbol{\beta}}_{i,t-1}\big)\Big)\right] \\
&= \frac{1}{r}\left[\sum_{j=1}^{r} Y_{ij} - \sum_{\ell=2}^{t-1} \sum_{j=r_{\ell-1}+1}^{r_\ell} f\big(\mathbf{C}_{ij}, \widehat{\boldsymbol{\beta}}_{i,\ell-1}\big) - \sum_{j=r_{t-1}+1}^{r} f\big(\mathbf{C}_{ij}, \widehat{\boldsymbol{\beta}}_{i,t-1}\big)\right].
\end{aligned}$$

Note that

$$\begin{aligned}
\big|\bar{\theta}_i(r) - \theta_i\big| &\leq \left|\frac{1}{r}\sum_{j=1}^{r}(Y_{ij} - \theta_i)\right| + \frac{1}{r}\left|\sum_{\ell=2}^{t-1} \sum_{j=r_{\ell-1}+1}^{r_\ell} f\big(\mathbf{C}_{ij}, \widehat{\boldsymbol{\beta}}_{i,\ell-1}\big) + \sum_{j=r_{t-1}+1}^{r} f\big(\mathbf{C}_{ij}, \widehat{\boldsymbol{\beta}}_{i,t-1}\big)\right| \\
&\leq \left|\frac{1}{r}\sum_{j=1}^{r}(Y_{ij} - \theta_i)\right| + \frac{r - n_0}{r} \times \sup_{\widehat{\boldsymbol{\beta}}_i \in \mathbf{B}} \left|\frac{1}{r - n_0} \sum_{j=n_0+1}^{r} f\big(\mathbf{C}_{ij}, \widehat{\boldsymbol{\beta}}_i\big)\right|. \qquad (11)
\end{aligned}$$

The first term in Equation (11) converges to 0 as $r \to 0$ by the conventional SLLN, and the second term converges to 0 by the assumption $\frac{n_0}{r} \to 0$ and Proposition 5.1 in Kim and Henderson (2007). □

As an immediate result of Lemma 1, we have the SLLN for the sample variance estimator.

Lemma 2 (Convergence of the Sample Variance Estimator)**.** *Suppose that all assumptions in Lemma 1 are satisfied. The updated sample variance estimator for system* i *is given in Equation* (4)*. Then, as* $r \to \infty$*,* $S_i^2(r) \xrightarrow{a.s.} \xi_i^2$*, where* $\xi_i^2 = (1 - R_i^2)\sigma_i^2$ *in the linear CV case and* $\xi_i^2 = v(\boldsymbol{\beta}_i^*)$ *in the nonlinear SAA case.*

Proof. Note that

$$\begin{aligned} S_i^2(r) &= \frac{1}{r-1}\Big[\sum_{j=1}^{r}\big(\widehat{\theta}_{ij}\big)^2 - \frac{1}{r}\Big(\sum_{j=1}^{r}\widehat{\theta}_{ij}\Big)^2\Big] \\ &= \frac{1}{r-1}\Big[\sum_{j=1}^{r_1}\big(\widehat{\theta}_{ij}\big)^2 + \sum_{j=r_1+1}^{r_2}\big(\widehat{\theta}_{ij}\big)^2 + \cdots + \sum_{j=r_{t-1}+1}^{r}\big(\widehat{\theta}_{ij}\big)^2 \\ &\qquad -\frac{1}{r}\Big(\sum_{j=1}^{r_1}\widehat{\theta}_{ij} + \sum_{j=r_1+1}^{r_2}\widehat{\theta}_{ij} + \cdots + \sum_{j=r_{t-1}+1}^{r}\widehat{\theta}_{ij}\Big)^2\Big]. \end{aligned} \tag{12}$$

Notice that, by the SLLN in Lemma 1 and the Continuous Mapping Theorem, $\frac{1}{r_1}\sum_{j=1}^{r_1}\widehat{\theta}_{ij} = \frac{1}{n_0}\sum_{j=1}^{n_0}Y_{ij} \xrightarrow{\text{a.s.}} \theta_i$ and $\frac{1}{r_1}\sum_{j=1}^{r_1}\big(\widehat{\theta}_{ij}\big)^2 = \frac{1}{n_0}\sum_{j=1}^{n_0}Y_{ij}^2 \xrightarrow{\text{a.s.}} \theta_i^2+\sigma_i^2$, as $n_0\to\infty$, where θ_i and σ_i^2 are the true mean and true variance of Y_{ij}. Since that $n_0\to\infty$, we know that $\widehat{\boldsymbol{\beta}}_{i,\ell-1} \xrightarrow{\text{a.s.}} \boldsymbol{\beta}_i^*$ for all $\ell\geq 2$. Then, for all $j>n_0$, $\widehat{\theta}_{ij}$ becomes i.i.d. with mean θ_i and variance ξ_i^2, where $\xi_i^2=(1-R_i^2)\sigma_i^2$ in the linear case and $\xi_i^2=v(\boldsymbol{\beta}_i^*)$ in the nonlinear SAA case, and $\big(\widehat{\theta}_{ij}\big)^2$ becomes i.i.d. with mean $\theta_i^2+\xi_i^2$. Then, by the Continuous Mapping Theorem again, $S_i^2(r)$ in Equation (12) can be further written as

$$\begin{aligned} S_i^2(r) &= \frac{1}{r-1}\Big[n_0\cdot\frac{1}{n_0}\sum_{j=1}^{n_0}\big(\widehat{\theta}_{ij}\big)^2 + \sum_{j=n_0+1}^{r}\big(\widehat{\theta}_{ij}\big)^2 - \frac{1}{r}\Big(n_0\cdot\frac{1}{n_0}\sum_{j=1}^{n_0}\widehat{\theta}_{ij} + \sum_{j=n_0+1}^{r}\widehat{\theta}_{ij}\Big)^2\Big] \\ &\xrightarrow{\text{a.s.}} \frac{1}{r-1}\Big[n_0(\theta_i^2+\sigma_i^2) + (r-n_0)(\theta_i^2+\xi_i^2) - \frac{1}{r}\big(n_0\theta_i+(r-n_0)\theta_i\big)^2\Big] \\ &= \frac{r}{r-1}\xi_i^2 + \frac{n_0}{r-1}(\sigma_i^2-\xi_i^2) \\ &\xrightarrow{\text{a.s.}} \xi_i^2, \end{aligned}$$

where the last equation is due to the fact $\frac{n_0}{r}\to 0$ as $r\to\infty$, which implies $\frac{r}{r-1}\to 1$ and $\frac{n_0}{r-1}\to 0$. □

Now, we are ready to introduce the scaled process. In fact, we are only interested in the pairwise comparisons between the true best system and all others, i.e., systems 1 and h, $h=2,\ldots,k$. Therefore, we intend to establish the convergence to a Brownian motion process for the statistics associated with $\bar{\theta}_1(r)-\bar{\theta}_h(r)$. In order to do so, we need to define the maximum number of samples taken from the pair of systems 1 and h to be $N_{1h}(\delta)=\lceil 2a(\xi_1^2+\xi_h^2)/\delta^2\rceil$, where $a=-\log[2\alpha/(k-1)]$ and $\xi_i^2=(1-R_i^2)\sigma_i^2$ in the linear case or $\xi_i^2=v(\boldsymbol{\beta}_i^*)$ in the nonlinear SAA case. For notational simplification, we denote $N_{1h}(\delta)$ by N, which is a function of the IZ parameter δ. We then consider the asymptotic regime that $\delta\to 0$, which has been studied in Kim and Nelson (2006a) and Luo et al. (2015).

Let $\epsilon=\frac{n_0}{N}<1$, which is also a function of δ. Recall that $n_0\to\infty$ and $\delta n_0\to 0$ as $\delta\to 0$. Then, we have $N\to\infty$ and $\epsilon\to 0$ as $\delta\to 0$. Let s be any number in $[0,1]$. Then, the sample size r can

be expressed as $r = \lfloor sN \rfloor$, where $\lfloor x \rfloor$ denotes the largest integer that is smaller than or equal to x. When $s \in [\epsilon, 1]$, we define the scaled statistics associated with $\bar{\theta}_1(r) - \bar{\theta}_h(r)$ as follows,

$$Z_{1h}(s) = \frac{1}{\sqrt{\xi_1^2 + \xi_h^2}} \cdot \frac{r}{\sqrt{N}} \Big[\bar{\theta}_1(r) - \bar{\theta}_h(r) \Big] = \frac{1}{\sqrt{\xi_1^2 + \xi_h^2}} \cdot \frac{1}{\sqrt{N}} \left[\sum_{j=1}^{\lfloor sN \rfloor} \widehat{\theta}_{1j} - \sum_{j=1}^{\lfloor sN \rfloor} \widehat{\theta}_{hj} \right]. \tag{13}$$

The reason that we only consider the time interval $[\epsilon, 1]$ is due to the fact that the comparison is conducted only when $r \geq n_0$, where the initial sample size $n_0 \geq 2$. However, for mathematical completeness, we also define $Z_{1h}(s) = 0$ when $s \in [0, \epsilon)$. Now we are ready to state the convergence of the scaled process.

Lemma 3 (Convergence of the Scaled Process)**.** *Let $\mathbb{D}[0,1]$ be the Skorohod space of all Càdlàg (i.e., right continuous with left limits) functions on $[0,1]$, endowed with the Skorohod J_1 topology. Then, $Z_{1h}(\cdot)$ defined by Equation (13) with $h = 2, 3, \ldots, k$ is an element of the Skorohod space $\mathbb{D}[0,1]$. Suppose that the conditions in Theorems 1 and 2 are all satisfied. Then, in the slippage configuration (i.e., $\theta_2 = \theta_3 = \cdots = \theta_k = \theta_1 - \delta$), we have*

$$Z_{1h}(\cdot) \Rightarrow \mathcal{B}_{\Delta}(\cdot), \quad \text{as } \delta \to 0,$$

where $\mathcal{B}_{\Delta}(s) = \mathcal{B}(s) + \Delta s$ is a standard Brownian motion process with a positive drift $\Delta = \sqrt{2a}$.

Proof. Recall the definition of $Z_{1h}(s) = 0$ when $s \in [0, \epsilon)$, and that $\epsilon \to 0$ as $\delta \to 0$, which means that $Z_{1h}(0) = \mathcal{B}_{\Delta}(0) = 0$ and $Z_{1h}(\cdot)$ is right-continuous at $s = 0$ for all δ. Then, in the following, we only focus on the interval $s \in [\epsilon, 1]$.

Ignoring the normalized term $\frac{1}{\sqrt{\xi_1^2+\xi_h^2}}$ in Equation (13), we start by investigating on the term $\frac{1}{\sqrt{N}} \sum_{j=1}^{\lfloor sN \rfloor} \Big(\widehat{\theta}_{ij} - \theta_i \Big)$, where $i = 1$ or h. Plugging Equation (10), we then have

$$\begin{aligned}
\frac{1}{\sqrt{N}} \sum_{j=1}^{\lfloor sN \rfloor} \Big(\widehat{\theta}_{ij} - \theta_i \Big) &= \frac{1}{\sqrt{N}} \sum_{j=1}^{\lfloor sN \rfloor} \Big(Y_{ij} - f\big(\mathbf{C}_{ij}, \widehat{\boldsymbol{\beta}}_{i,\ell-1}\big) - \theta_i \Big), \\
&= \frac{1}{\sqrt{N}} \left[\sum_{j=1}^{n_0} (Y_{ij} - \theta_i) + \sum_{j=n_0+1}^{\lfloor sN \rfloor} \Big(Y_{ij} - f\big(\mathbf{C}_{ij}, \widehat{\boldsymbol{\beta}}_{i,\ell-1}\big) - \theta_i \Big) \right] \\
&= \frac{1}{\sqrt{N}} \left[\sum_{j=1}^{n_0} f\big(\mathbf{C}_{ij}, \boldsymbol{\beta}_i^*\big) + \sum_{j=1}^{n_0} \big(Y_{ij} - f\big(\mathbf{C}_{ij}, \boldsymbol{\beta}_i^*\big) - \theta_i \big) + \sum_{j=n_0+1}^{\lfloor sN \rfloor} \big(Y_{ij} - f\big(\mathbf{C}_{ij}, \boldsymbol{\beta}_i^*\big) - \theta_i \big) \right. \\
&\quad \left. + \sum_{j=n_0+1}^{\lfloor sN \rfloor} \Big(f\big(\mathbf{C}_{ij}, \boldsymbol{\beta}_i^*\big) - f\big(\mathbf{C}_{ij}, \widehat{\boldsymbol{\beta}}_{i,\ell-1}\big) \Big) \right].
\end{aligned} \tag{14}$$

Due to the fact that $r_1 = n_0$ and the assumption that $n_0 \to \infty$ as $\delta \to 0$, then we know that $\widehat{\boldsymbol{\beta}}_{i,\ell-1} \xrightarrow{\text{a.s.}} \boldsymbol{\beta}_i^*$ for all $\ell \geq 2$.[13] Recall that $N = \lceil 2a(\xi_1^2 + \xi_h^2)/\delta^2 \rceil$, which converges to ∞ as $\delta \to 0$. Moreover, the assumption that $\delta n_0 \to 0$ as $\delta \to 0$ implies not only $\epsilon = \frac{n_0}{N} \to 0$, but also $\frac{n_0}{\sqrt{N}} \to 0$. By Proposition 5.1 in Kim and Henderson (2007) again, we know that $\frac{1}{n_0} \sum_{j=1}^{n_0} f\big(\mathbf{C}_{ij}, \boldsymbol{\beta}_i^*\big) \xrightarrow{\text{a.s.}} 0$. Then, as $\delta \to 0$, the first term in Equation (14)

$$\frac{1}{\sqrt{N}} \sum_{j=1}^{n_0} f\big(\mathbf{C}_{ij}, \boldsymbol{\beta}_i^*\big) = \frac{n_0}{\sqrt{N}} \cdot \frac{1}{n_0} \sum_{j=1}^{n_0} f\big(\mathbf{C}_{ij}, \boldsymbol{\beta}_i^*\big) \xrightarrow{\text{a.s.}} 0.$$

For the last term in Equation (14), it suffices to show that

$$D = \frac{1}{r - n_0} \sum_{j=n_0+1}^{r} \Big(f\big(\mathbf{C}_{ij}, \boldsymbol{\beta}_i^*\big) - f\big(\mathbf{C}_{ij}, \widehat{\boldsymbol{\beta}}_{i,\ell-1}\big) \Big) \Rightarrow 0.$$

By Chebyshev's inequality, we know that, for any given $\epsilon > 0$,

$$\Pr\{|D| \geq \epsilon\} \leq \frac{1}{(r - n_0)\epsilon^2} \sum_{j=n_0+1}^{r} \mathrm{E}\big[f\big(\mathbf{C}_{ij}, \boldsymbol{\beta}_i^*\big) - f\big(\mathbf{C}_{ij}, \widehat{\boldsymbol{\beta}}_{i,\ell-1}\big) \big]^2 \tag{15}$$

Note that,

$$\big[f\big(\mathbf{C}_{ij}, \boldsymbol{\beta}_i^*\big) - f\big(\mathbf{C}_{ij}, \widehat{\boldsymbol{\beta}}_{i,\ell-1}\big) \big]^2 \leq W^2(\mathbf{C}_{ij}) \|\boldsymbol{\beta}_i^* - \widehat{\boldsymbol{\beta}}_{i,\ell-1}\|^2 \tag{16}$$

The normed term in (16) is bounded, so the dominated convergence theorem (cf., Billingsley (1986)) implies that (15) converges to 0 as $r \to \infty$.

For the second and third term in Equation (14), they can be combined together, and $\{Y_{ij} - f\big(\mathbf{C}_{ij}, \boldsymbol{\beta}_i^*\big) - \theta_i, j = 1, 2, \ldots\}$ are i.i.d. random variables with mean zero and variance $\xi_i^2 = (1 - R_i^2)\sigma_i^2$ or $v(\boldsymbol{\beta}_i^*)$. Then, by the Donsker's Theorem (i.e., Theorem 4.3.2 in Whitt (2002)),

$$\frac{1}{\sqrt{N}} \left[\sum_{j=1}^{\lfloor sN \rfloor} \big(Y_{ij} - f\big(\mathbf{C}_{ij}, \boldsymbol{\beta}_i^*\big) - \theta_i \big) \right] \Rightarrow \xi_i \mathcal{B}^{(i)}(s),$$

where $\mathcal{B}^{(i)}(\cdot)$ is a standard Brownian motion process.

[13] We do not need the assumption of $\widehat{\boldsymbol{\beta}}_i(r) \xrightarrow{\text{a.s.}} \boldsymbol{\beta}_i^*$ for a constant $\boldsymbol{\beta}_i^*$ under the linear CV approach. It should be noticed that, by Theorem 3 in Nelson (1990), we know that $\widehat{\boldsymbol{\beta}}_i(r_1) \xrightarrow{\text{p}} \boldsymbol{\beta}_i^*$ as $\delta \to 0$. There is a stronger result $\widehat{\boldsymbol{\beta}}_i(r) \xrightarrow{\text{a.s.}} \boldsymbol{\beta}_i^*$ as $r \to \infty$ (cf., Avramidis and Wilson (1993)). In fact, the convergence in probability result in Nelson (1990) is sufficient for our proof.

Then, we are ready to conclude this proof by showing that

$$\begin{aligned} Z_{1h}(s) &= \frac{1}{\sqrt{\xi_1^2+\xi_h^2}} \cdot \frac{1}{\sqrt{N}} \left[\sum_{j=1}^{\lfloor sN \rfloor} \left(\widehat{\theta}_{1j} - \theta_1 \right) - \sum_{j=1}^{\lfloor sN \rfloor} \left(\widehat{\theta}_{hj} - \theta_h \right) \right] + \frac{1}{\sqrt{\xi_1^2+\xi_h^2}} \cdot \frac{1}{\sqrt{N}} \Big[\lfloor sN \rfloor (\theta_1 - \theta_h) \Big] \\ &\Rightarrow \frac{1}{\sqrt{\xi_1^2+\xi_h^2}} \cdot \Big[\xi_1 \mathcal{B}^{(1)}(s) + \xi_h \mathcal{B}^{(h)}(s) \Big] + \Delta s, \text{ where } \Delta = \sqrt{2a}, \\ &= \mathcal{B}(s) + \Delta s \stackrel{\mathcal{D}}{=} \mathcal{B}_\Delta(s), \end{aligned}$$

where the last equation is due to the independence among $\mathcal{B}^{(1)}(s)$ and $\mathcal{B}^{(h)}(s)$, and the notation $\stackrel{\mathcal{D}}{=}$ means "equal in distribution". □

In Lemma 3, we construct the continuous-time stochastic process $Z_{1h}(s)$ by properly scaling the original discrete-time stochastic process $\bar{\theta}_1(r) - \bar{\theta}_h(r)$ by the term $\frac{1}{\sqrt{\xi_1^2+\xi_h^2}} \cdot \frac{r}{\sqrt{N}}$, in order to establish the limiting Brownian motion process $\mathcal{B}_\Delta(s)$. However, in order to show the asymptotic statistical guarantee, it requires to scale not only the original statistics (i.e., the term in the left-hand-side (LHS) of Inequality (9)), but also the continuation region for elimination (i.e., the term in the right-hand-side (RHS) of Inequality (9)).

For the sake of presentational simplicity, we define the upper bound and lower bound as follows,

$$\Gamma_{ih}(r) = \max\left\{0, \frac{a}{\delta} \cdot \frac{[S_i^2(r) + S_h^2(r)]}{r} - \frac{\delta}{2}\right\} \text{ and } -\Gamma_{ih}(r) = \min\left\{0, -\frac{a}{\delta} \cdot \frac{[S_i^2(r) + S_h^2(r)]}{r} + \frac{\delta}{2}\right\},$$

which forms the symmetric continuation region Λ_{ih} for the pair of systems i and h. Since we are interested in the pair of systems 1 and h, where $h = 2, 3, \ldots, k$, we next consider only the continuation region Λ_{1h}, for $h = 2, 3, \ldots, k$.

To make the comparison condition also hold for $Z_{1h}(s)$, we scale the RHS in Inequality (9) by the same term, which is done as follows. Define

$$\Gamma_{1h}^{\delta}(s) = \frac{1}{\sqrt{\xi_1^2+\xi_h^2}} \cdot \frac{r}{\sqrt{N}} \cdot \Gamma_{1h}(r) = \max\left\{0, \frac{a\left[S_1^2(r) + S_h^2(r)\right]}{\delta\sqrt{\xi_1^2+\xi_h^2}\sqrt{N}} - \frac{\delta r}{2\sqrt{\xi_1^2+\xi_h^2}\sqrt{N}}\right\}. \tag{17}$$

Then, the upper boundary $\Gamma_{1h}^{\delta}(s)$ and lower boundary $-\Gamma_{1h}^{\delta}(s)$ form the symmetric continuation region Λ_{1h}^{δ} for the scaled process $Z_{1h}(s)$. Notice that either system 1 or h is eliminated depends on whether $Z_{1h}(\cdot)$ exits the continuation region Λ_{1h}^{δ} from above or below. As $\delta \to 0$, we expect the scaled continuation region Λ_{1h}^{δ} also converges to some limiting continuation region, which is rigorously stated in the following lemma.

Lemma 4 (Convergence of the Triangular Region)**.** *Define the symmetric continuation region* Λ_{1h}^{δ}

for $Z_{1h}(\cdot)$ by the upper boundary $\Gamma^{\delta}_{1h}(s)$ and lower boundary $-\Gamma^{\delta}_{1h}(s)$ as in Equation (17)*. Suppose that the conditions in Theorems 1 and 2 are all satisfied. Then, as $\delta \to 0$, we know that*

$$\Gamma^{\delta}_{1h}(s) \xrightarrow{a.s.} \Gamma(s) = \max\left\{0, \frac{a}{\Delta} - \frac{\Delta}{2} \cdot s\right\}, \text{ for all } s \in (0, 1],$$

where $\Delta = \sqrt{2a}$. Moreover, the asymptotic region Λ, formed by $\Gamma(s)$ and $-\Gamma(s)$, is a symmetric triangular region for the Brownian motion process $\mathcal{B}_{\Delta}(\cdot)$ as obtained in Lemma 3.

Remark 5. *The reason we exclude the case that $s = 0$ is simply because the sample variance estimator $S^2_i(r)$ in Equation* (17) *is only defined when $r \geq n_0$, which implies $s > 0$. For mathematical rigorousness, we may define $S^2_i(r) = 0$ for $r < n_0$, which then allows us to include the original point $s = 0$.*

Proof. Recall the definition of r and N, which are $r = \lfloor sN \rfloor$ and $N = \lceil 2a(\xi_1^2 + \xi_h^2)/\delta^2 \rceil$, where $r \geq n_0$ implies $s > 0$. It is easy to show that

$$\frac{a}{\delta\sqrt{\xi_1^2 + \xi_h^2}\sqrt{N}} \to \frac{a}{\Delta(\xi_1^2 + \xi_h^2)}, \quad \text{and} \quad \frac{\delta r}{2\sqrt{\xi_1^2 + \xi_h^2}\sqrt{N}} \to \frac{\Delta}{2} \cdot s,$$

as $\delta \to 0$. Then, the key step in proving Lemma 4 is to show the convergence of the sample variance estimator $S^2_i(r) \xrightarrow{\text{a.s.}} \xi_i^2$ in Equation (17), which has been shown in Lemma 2. □

Notice that Lemmas 3 and 4 have established the convergence results of the scaled stochastic process $Z_{1h}(\cdot)$ and the scaled continuation region Λ^{δ}_{1h}. However, recall that elimination decisions are only made at these stopping times when the stochastic processes first exit the continuation regions, which indicates that we need to obtain a stronger result to ensure the convergence at these stopping times in order to bound the probability of incorrect selection.

Lemma 5 (Convergence at the Stopping Time)**.** *Define $T^{\delta}_{1h} = \inf\left\{s : |Z_{1h}(s)| \geq \Gamma^{\delta}_{1h}(s)\right\}$ as the stopping time when $Z_{1h}(\cdot)$ first exits the continuation region Λ^{δ}_{1h} and define $T_{1h} = \inf\{s : |\mathcal{B}_{\Delta}(s)| \geq \Gamma(s)\}$ as the stopping time when $\mathcal{B}_{\Delta}(\cdot)$ first exits the triangular region Λ. Suppose that the conditions in Theorems 1 and 2 are all satisfied. Then, as $\delta \to 0$, we have*

$$Z_{1h}(T^{\delta}_{1h}) \Rightarrow \mathcal{B}_{\Delta}(T_{1h}).$$

Remark 6. *We omit the proof of Lemma 5, because it follows exactly the same steps as that in proving Proposition 3.2 of Kim et al. (2005), Lemma 2 in Luo et al. (2015) and Lemma 3 in Tsai et al. (2017).*

We also need the lemma of Fabian (1974) on the probability of $\mathcal{B}_\Delta(\cdot)$ exiting the triangular continuation region Λ.

Lemma 6 (Fabian's Bound at the Stopping Time)**.** *For any fixed triangular continuation region, which is formed by the upper boundary* $U(s) = \max\{0, A - Bs\}$ *and the lower boundary* $-U(s)$*, if* $B = \Delta/2$ *where* $\Delta > 0$*, then*

$$\Pr\{\mathcal{B}_\Delta(T) < 0\} = \frac{1}{2}e^{-A\Delta},$$

where $\mathcal{B}_\Delta(\cdot)$ *is the standard Brownian motion process with the positive drift term* Δ*, and* $T = \inf\{s : |\mathcal{B}_\Delta(s)| \geq U(s)\}$ *is the random stopping time that* $\mathcal{B}_\Delta(\cdot)$ *first exits the continuation region.*

We are now ready to prove Theorems 1 and 2.

Proof. Recall our objective is to show that the limiting PCS level (i.e., the probability of selecting system 1) satisfies that

$$\liminf_{\delta \to 0} \Pr\{\text{select system 1}\} \geq 1 - \alpha. \tag{18}$$

By Bonferroni inequality (cf., Kim and Nelson (2006b)), the LHS of Equation (18) can be relaxed as follows,

$$\begin{aligned}
\liminf_{\delta \to 0} \Pr\{\text{select system 1}\} &= \liminf_{\delta \to 0} \left[1 - \Pr\left\{\bigcup_{h=2}^{k} \{\text{system } h \text{ eliminates 1}\}\right\}\right] \\
&\geq 1 - \limsup_{\delta \to 0} \sum_{h=2}^{k} \Pr\{\text{system } h \text{ eliminates 1}\}, \qquad (19)
\end{aligned}$$

which allows us to deal with the term $\limsup_{\delta \to 0} \Pr\{\text{system } h \text{ eliminates 1}\}$, i.e., the limiting probability of incorrect selection that system 1 is eliminated by system h, where $h = 2, 3, \ldots, k$.

We begin with the slippage configuration where $\theta_1 - \delta = \theta_2 = \cdots = \theta_k$. Notice that the

$$\begin{aligned}
\limsup_{\delta \to 0} \Pr\{\text{system } h \text{ eliminates 1}\} &= \limsup_{\delta \to 0} \Pr\left\{Z_{1h}\left(T_{1h}^{\delta}\right) \leq 0\right\} && (20) \\
&= \Pr\{\mathcal{B}_\Delta(T_{1h}) \leq 0\} && (21) \\
&= \frac{1}{2}e^{-\frac{a}{\Delta}\Delta} && (22) \\
&= \frac{\alpha}{k-1}, && (23)
\end{aligned}$$

where Equation (20) holds because of the fact that $Z_{1h}(\cdot)$ exits the continuation region through the lower boundary, Equation (21) follows from Lemma 5, Equation (22) follows from Lemma 6, and

Equation (23) holds because of the definition of a and Δ. Plugging (23) into (19), we achieve our objective by obtaining that

$$\liminf_{\delta\to 0} \Pr\{\text{select system } 1\} \;\geq\; 1-\sum_{h=2}^{k}\frac{\alpha}{k-1} = 1-\alpha.$$

We next deal with the general cases under the IZ formulation, i.e., $\theta_1-\delta\geq\theta_2\geq\cdots\geq\theta_k$. In that situation, the convergence of the scaled stochastic process $Z_{1h}(\cdot)$ to the Brownian motion process $\mathcal{B}_\Delta(\cdot)$ is no longer true. However, we can define a new type of scaled process

$$\widehat{Z}_{1h}(s) \;=\; \frac{1}{\sqrt{\xi_1^2+\xi_h^2}}\cdot\frac{r}{\sqrt{N}}\big[\,\big(\bar{\theta}_1(r)-\bar{\theta}_h(r)\big)-(\theta_1-\theta_h-\delta)\,\big].$$

By Lemma 3, we know that $\widehat{Z}_{1h}(\cdot)\Rightarrow\mathcal{B}_\Delta(\cdot)$ as $\delta\to 0$. Moreover, because of the IZ assumption that $\theta_1-\delta\geq\theta_h$, for all $h=2,3,\ldots,k$, we have that

$$\widehat{Z}_{1h}(s) = Z_{1h}(s)-\frac{1}{\sqrt{\xi_1^2+\xi_h^2}}\cdot\frac{r}{\sqrt{N}}(\theta_1-\theta_h-\delta)\leq Z_{1h}(s). \tag{24}$$

Define $\widehat{T}_{1h}^{\delta}$ as the stopping time at which $\widehat{Z}_{1h}(\cdot)$ first exits the continuation region Λ_{1h}^{δ}, i.e.,

$$\widehat{T}_{1h}^{\delta} = \inf\left\{s:\left|\widehat{Z}_{1h}(s)\right|\geq\Gamma_{1h}^{\delta}(s)\right\}.$$

Then, using the result in Inequality (24), we can bound the limiting probability of incorrect selection between systems 1 and h as follows,

$$\begin{aligned}
\limsup_{\delta\to 0}\ \Pr\{\text{system } h \text{ eliminates } 1\} &= \limsup_{\delta\to 0}\ \Pr\left\{Z_{1h}\left(T_{1h}^{\delta}\right)\leq 0\right\}\\
&\leq \limsup_{\delta\to 0}\ \Pr\left\{\widehat{Z}_{1h}\left(\widehat{T}_{1h}^{\delta}\right)\leq 0\right\}\\
&= \Pr\{\mathcal{B}_\Delta(T_{1h})\leq 0\}\\
&= \frac{\alpha}{k-1}.
\end{aligned} \tag{25}$$

Again, plugging (25) into (19), we conclude the proof. □

Remark 7. *(a) It is interesting to point out that the scaled continuation region $\Gamma_{1h}^{\delta}(\cdot)$ is the same for both the slippage configuration and general IZ cases, which allows us to define the stopping times T_{1h}^{δ} or $\widehat{T}_{1h}^{\delta}$ for either $Z_{1h}(\cdot)$ or $\widehat{Z}_{1h}(\cdot)$ using the same continuation region.*

(b) It is also worthwhile noticing that, as $Z_{1h}(\cdot)$ exits the continuation region $\Gamma_{1h}^{\delta}(\cdot)$ through the

lower boundary at time T_{1h}^{δ}*, i.e.,* $Z_{1h}\left(T_{1h}^{\delta}\right) \leq 0$*, Inequality* (24) *implies that* $\widehat{Z}_{1h}(\cdot)$ *must have already exited the same continuation region* $\Gamma_{1h}^{\delta}(\cdot)$ *through the lower boundary at an earlier time* $\widehat{T}_{1h}^{\delta}$*, i.e.,* $\widehat{Z}_{1h}\left(\widehat{T}_{1h}^{\delta}\right) \leq 0$*, with* $\widehat{T}_{1h}^{\delta} < T_{1h}^{\delta}$*.*

B Proofs of Propositions 3 and 4

In this section, we provide the CLT of $\bar{\theta}_i(r)$ under two different types of assumptions. The first type of assumption has the same setting as Theorem 2, where $\delta \to 0$ implies that $n_0 \to \infty$. We state the result in Proposition 3. The second type of assumption is the sequence dependent assumption, i.e., assuming that the sequence $\left\{\widehat{\theta}_{ij}, j \in \mathbb{Z}^+\right\}$ is strictly stationary. Under the sequence dependent assumption, we do not need n_0 goes to infinity. We state the result in Proposition 4.

Proposition 3. *Suppose that Assumptions 1–4 hold. Let the first-stage sample size* $n_0 = n_0(\delta)$ *be a function of* δ*, satisfying that* $n_0(\delta) \geq 2$ *and that* $n_0 \to \infty$ *and* $\delta n_0 \to 0$ *as* $\delta \to 0$*. Let the total sample size* $r = r(\delta)$ *also be a function of* δ*, satisfying that* $r > n_0$ *and* $\delta^2 r \to c$ *where* $c > 0$ *is a positive constant. Then the SAA-based nonlinear CV estimator* $\bar{\theta}_i(r)$ *in Equation* (4)*, satisfies that, as* $\delta \to 0$*,*

$$\frac{\sqrt{r}\big(\bar{\theta}_i(r) - \theta_i\big)}{\xi_i} \Rightarrow N(0,1),$$

where $\xi_i^2 = v(\boldsymbol{\beta}_i^*)$*.*

The proof of Proposition 3 is quite similar to the proof of the convergence to a Brownian motion process in Lemma 3.

Proof. At any stage $t \geq 2$ with $r > n_0$, we know that

$$\begin{aligned}
\frac{1}{\sqrt{r}}\sum_{j=1}^{r}\left(\widehat{\theta}_{ij} - \theta_i\right) &= \frac{1}{\sqrt{r}}\left[\sum_{j=1}^{n_0}\left(\widehat{\theta}_{ij} - \theta_i\right) + \sum_{\ell=2}^{t-1}\sum_{j=r_{\ell-1}+1}^{r_\ell}\left(\widehat{\theta}_{ij} - \theta_i\right) + \sum_{j=r_{t-1}+1}^{r}\left(\widehat{\theta}_{ij} - \theta_i\right)\right] \\
&= \frac{1}{\sqrt{r}}\left[\sum_{j=1}^{n_0}\left(Y_{ij} - \theta_i\right) + \sum_{j=n_0+1}^{r}\left(Y_{ij} - f\big(\mathbf{C}_{ij}, \widehat{\boldsymbol{\beta}}_{i,\ell-1}\big) - \theta_i\right)\right] \\
&= \frac{1}{\sqrt{r}}\left[\sum_{j=1}^{n_0} f\big(\mathbf{C}_{ij}, \boldsymbol{\beta}_i^*\big) + \sum_{j=1}^{n_0}\big(Y_{ij} - f\big(\mathbf{C}_{ij}, \boldsymbol{\beta}_i^*\big) - \theta_i\big)\right. \\
&\quad + \sum_{j=n_0+1}^{r}\big(Y_{ij} - f\big(\mathbf{C}_{ij}, \boldsymbol{\beta}_i^*\big) - \theta_i\big) \\
&\quad \left. + \sum_{j=n_0+1}^{r}\Big(f\big(\mathbf{C}_{ij}, \boldsymbol{\beta}_i^*\big) - f\big(\mathbf{C}_{ij}, \widehat{\boldsymbol{\beta}}_{i,\ell-1}\big)\Big)\right]. \qquad (26)
\end{aligned}$$

Due to the fact that $r_1 = n_0$ and the assumption that $n_0 \to \infty$ as $\delta \to 0$, then we know that $\widehat{\boldsymbol{\beta}}_{i,\ell-1} \xrightarrow{\text{a.s.}} \boldsymbol{\beta}_i^*$ for all $\ell \geq 2$. By Proposition 5.1 in Kim and Henderson (2007) again, we know that $\frac{1}{n_0}\sum_{j=1}^{n_0} f\big(\mathbf{C}_{ij}, \boldsymbol{\beta}_i^*\big) \xrightarrow{\text{a.s.}} 0$. Recall that $\delta n_0 \to 0$ and $\delta^2 r \to c$. Then, as $\delta \to 0$, the first term in Equation (26)

$$\frac{1}{\sqrt{r}}\sum_{j=1}^{n_0} f\big(\mathbf{C}_{ij}, \boldsymbol{\beta}_i^*\big) = \frac{n_0}{\sqrt{r}} \cdot \frac{1}{n_0}\sum_{j=1}^{n_0} f\big(\mathbf{C}_{ij}, \boldsymbol{\beta}_i^*\big) \xrightarrow{\text{a.s.}} 0.$$

For the last term in Equation (26), it suffices to show that

$$D = \frac{1}{r-n_0}\sum_{j=n_0+1}^{r} \Big(f\big(\mathbf{C}_{ij}, \boldsymbol{\beta}_i^*\big) - f\big(\mathbf{C}_{ij}, \widehat{\boldsymbol{\beta}}_{i,\ell-1}\big)\Big) \Rightarrow 0.$$

By Chebyshev's inequality, we know that, for any given $\epsilon > 0$,

$$\Pr\{|D| \geq \epsilon\} \leq \frac{1}{(r-n_0)\epsilon^2}\sum_{j=n_0+1}^{r} \mathrm{E}\big[f\big(\mathbf{C}_{ij}, \boldsymbol{\beta}_i^*\big) - f\big(\mathbf{C}_{ij}, \widehat{\boldsymbol{\beta}}_{i,\ell-1}\big)\big]^2 \tag{27}$$

Note that,

$$\big[f\big(\mathbf{C}_{ij}, \boldsymbol{\beta}_i^*\big) - f\big(\mathbf{C}_{ij}, \widehat{\boldsymbol{\beta}}_{i,\ell-1}\big)\big]^2 \leq W^2(\mathbf{C}_{ij})\|\boldsymbol{\beta}_i^* - \widehat{\boldsymbol{\beta}}_{i,\ell-1}\|^2 \tag{28}$$

The normed term in (28) is bounded, so the dominated convergence theorem (cf., Billingsley (1986)) implies that (27) converges to 0 as $r \to \infty$.

For the second and third term in Equation (26), they can be combined together, and $\{Y_{ij} - f\big(\mathbf{C}_{ij}, \boldsymbol{\beta}_i^*\big) - \theta_i, j = 1, 2, \ldots\}$ are i.i.d. random variables with mean zero and variance $\xi_i^2 = v(\boldsymbol{\beta}_i^*)$. Then, by conventional CLT,

$$\frac{1}{\sqrt{r}}\left[\sum_{j=1}^{r}\big(Y_{ij} - f\big(\mathbf{C}_{ij}, \boldsymbol{\beta}_i^*\big) - \theta_i\big)\right] \Rightarrow \xi_i N(0,1).$$

Therefore, Proposition 3 has been proved. □

In the proof above, when $n_0 \to \infty$, $\widehat{\boldsymbol{\beta}}_{i,\ell-1} \to \widehat{\boldsymbol{\beta}}_i^*$, so the CV estimators in the following stages become independent. If we do not let $n_0 \to \infty$, we need to know further dependent property of the sequence $\big\{\widehat{\theta}_{ij}, j \in \mathbb{Z}^+\big\}$. To provide another type of CLT, we need to define a big-O notation: for a real valued function g, if there exists a positive real number M and a real number r_0 such that $|g(r)| \leq Mr$ for all $r \geq r_0$, we then write $g(r) = \mathcal{O}(r)$.

Proposition 4. *Suppose that Assumptions 1–4 hold. For any system i, $\{\widehat{\theta}_{ij}, j \in \mathbb{Z}^+\}$ is assumed to be a strictly stationary random variable sequence such that $\mathrm{E}\big[(\sum_{j=1}^{r}(\widehat{\theta}_{ij}-\theta_i))^2\big] = \mathcal{O}(r)$, and $0 < \mathrm{E}[\widehat{\theta}_{i1}^2] < \infty$. Then the SAA-based nonlinear CV estimator $\bar{\theta}_i(r)$ in Equation (4) satisfies that, as $r \to \infty$,*

(i) $\bar{\theta}_i(r) \xrightarrow{a.s.} \theta_i$.

(ii)

$$\frac{r\big(\bar{\theta}_i(r)-\theta_i\big)}{\sigma_{ir}} \Rightarrow N(0,1), \tag{29}$$

where $\sigma_{ir}^2 = r^2\mathrm{Var}\left[\bar{\theta}_i(r)\right] \to \infty$.

Remark 8. *Equation (29) is not a classical expression of CLT. The assumption that $\mathrm{E}[(\sum_{j=1}^{r}(\widehat{\theta}_{ij}-\theta_i))^2] = \mathcal{O}(r)$ implicates the growth of σ_{ir}^2. Notice that $\sigma_{ir}^2 = r^2\mathrm{Var}\left[\bar{\theta}_i(r)\right] = \mathrm{Var}[\sum_{j=1}^{r}\widehat{\theta}_{ij}]$, which can be shown to converge to infinity as $r \to \infty$, so both the numerator and the denominator go to infinity as $r \to \infty$. But if this variance can be expressed by $\mathrm{Var}[\sum_{j=1}^{r}\widehat{\theta}_{ij}] = r\tilde{\xi}_i^2$ (i.e., $\widehat{\theta}_{ij}, j = 1, 2, \ldots$ are mutually independent, and its variance is $\tilde{\xi}_i^2$), then the result in Equation (29) can be written as $\sqrt{r}(\bar{\theta}_i(r)-\theta_i)/\tilde{\xi}_i \Rightarrow N(0,1)$, which is reduced to the classical CLT.*

In the following, we first prove Proposition 4 (i), and then introduce several existing results to build the proof of Proposition 4 (ii).

Proof of Proposition 4 (i). We first prove the convergence of $\widehat{\boldsymbol{\beta}}_i(r)$. By Assumption 2, $f(\mathbf{C}_{ij}, \boldsymbol{\beta}_i)$ is continuous at $\boldsymbol{\beta}_i$ a.s., and by Assumption 1, $\mathrm{E}[Y_{ij}^2] < \infty$ and $\mathrm{E}[f^2(\mathbf{C}_{ij}, \boldsymbol{\beta}_i)] < \infty$ for all $\boldsymbol{\beta}_i \in \mathscr{U}$, so we know that $\mathrm{E}[(Y_{ij} - f(\mathbf{C}_{ij}, \boldsymbol{\beta}_i))^2] < \infty$. It implies that $\mathrm{E}[\sup_{\boldsymbol{\beta}_i \in \mathcal{B}(\delta_B, \boldsymbol{\beta}_i^*)}(Y_{ij} - f(\mathbf{C}_{ij}, \boldsymbol{\beta}_i))^2] < \infty$, so we have $\{(Y_{ij} - f(\mathbf{C}_{ij}, \boldsymbol{\beta}_i))^2, \boldsymbol{\beta}_i \in \mathcal{B}(\delta_B, \boldsymbol{\beta}_i^*)\}$ is uniformly integrable. By Proposition 8.5 of Kim et al. (2015), $\mathrm{Var}(\boldsymbol{\beta}_i, r)$ converges to $\mathrm{Var}\left[Y_{ij} - f(\mathbf{C}_{ij}, \boldsymbol{\beta}_i)\right]$ uniformly. Then by Theorem 8.2 of Kim et al. (2015), the first-order critical point $\widehat{\boldsymbol{\beta}}_i(r)$ for the minimization of $\mathrm{Var}(\boldsymbol{\beta}_i, r)$ converges (a.s.) to the first-order critical point $\boldsymbol{\beta}_i^*$ for the minimization of $\mathrm{Var}\left[Y_{ij} - f(\mathbf{C}_{ij}, \boldsymbol{\beta}_i)\right]$. Notice that when $r \to \infty$, $\ell \to \infty$, then by the definition of $\widehat{\boldsymbol{\beta}}_{i\ell}$, we have that $\widehat{\boldsymbol{\beta}}_{i\ell} \xrightarrow{\text{a.s.}} \boldsymbol{\beta}_i^*$.

Recall that the nonlinear CV estimator (used in $\mathcal{AFS}$-$\mathcal{SAA}$) is $\bar{\theta}_i(r) = 1/r\sum_{j=1}^{r}\widehat{\theta}_{ij}$, with $\widehat{\theta}_{ij} = Y_{ij} - f\big(\mathbf{C}_{ij}, \widehat{\boldsymbol{\beta}}_{i\ell}\big)$, where $\ell = \max\{0, \lceil (j-n_0)/L \rceil\}$. Notice that

$$\left|\bar{\theta}_i(r) - \theta_i\right| \le \left|\frac{1}{r}\sum_{j=1}^{r}(Y_{ij}-\theta_i)\right| + \left|\frac{1}{r}\sum_{j=1}^{r} f\big(\mathbf{C}_{ij}, \widehat{\boldsymbol{\beta}}_{i\ell}\big)\right|. \tag{30}$$

The first term in Equation (30) converges to 0 as $r \to \infty$ by the conventional SLLN.

Consider the second term in Equation (30). Notice that the $\widehat{\boldsymbol{\beta}}_{i\ell}$ depends on the previous samples, so we denote this dependence as $\widehat{\boldsymbol{\beta}}_{i\ell(j)}$, which $\widehat{\boldsymbol{\beta}}_{i\ell(j)} \xrightarrow{\text{a.s.}} \boldsymbol{\beta}_i^*$, so there exists a set $E_1 \in \mathcal{F}$ ($\mathcal{F}$ is the

σ-field of the probability space) with $\Pr\{E_1\} = 0$ such that $\forall \omega \in E_1^c$ and $\forall \epsilon > 0$, $\exists r_1 \in \mathbb{Z}^+$ such that $\forall r > r_1$,

$$\left| \widehat{\boldsymbol{\beta}}_{i\ell(j)} - \boldsymbol{\beta}_i^* \right| < \epsilon. \tag{31}$$

By the Taylor expansion, we have

$$f\left(\mathbf{C}_{ij}, \widehat{\boldsymbol{\beta}}_{i\ell(j)}\right) = f\left(\mathbf{C}_{ij}, \boldsymbol{\beta}_i^*\right) + \nabla_{\boldsymbol{\beta}} f\left(\mathbf{C}_{ij}, \tilde{\boldsymbol{\beta}}_{i\ell(j)}\right)^\top \left(\widehat{\boldsymbol{\beta}}_{i\ell(j)} - \boldsymbol{\beta}_i^*\right),$$

where $\tilde{\boldsymbol{\beta}}_{i\ell(j)}$ is in the line of $\boldsymbol{\beta}_i^*$ and $\widehat{\boldsymbol{\beta}}_{i\ell(j)}$.

Then,

$$\begin{aligned}
& \left| \frac{1}{r} \sum_{j=1}^{r} f\left(\mathbf{C}_{ij}, \widehat{\boldsymbol{\beta}}_{i\ell(j)}\right) \right| \\
= & \left| \frac{1}{r} \sum_{j=1}^{r_1} f\left(\mathbf{C}_{ij}, \widehat{\boldsymbol{\beta}}_{i\ell(j)}\right) + \frac{1}{r} \sum_{j=r_1+1}^{r} f\left(\mathbf{C}_{ij}, \widehat{\boldsymbol{\beta}}_{i\ell(j)}\right) \right| \\
= & \left| \frac{1}{r} \sum_{j=1}^{r_1} f\left(\mathbf{C}_{ij}, \widehat{\boldsymbol{\beta}}_{i\ell(j)}\right) + \frac{1}{r} \sum_{j=r_1+1}^{r} \left\{ f\left(\mathbf{C}_{ij}, \boldsymbol{\beta}_i^*\right) + \nabla_{\boldsymbol{\beta}} f\left(\mathbf{C}_{ij}, \tilde{\boldsymbol{\beta}}_{i\ell(j)}\right)^\top \left(\widehat{\boldsymbol{\beta}}_{i\ell(j)} - \boldsymbol{\beta}_i^*\right) \right\} \right| \\
\leq & \left| \frac{1}{r} \sum_{j=1}^{r_1} f\left(\mathbf{C}_{ij}, \widehat{\boldsymbol{\beta}}_{i\ell(j)}\right) \right| + \left| \frac{1}{r} \sum_{j=r_1+1}^{r} f\left(\mathbf{C}_{ij}, \boldsymbol{\beta}_i^*\right) \right| + \frac{1}{r} \sum_{j=r_1+1}^{r} \left| \nabla_{\boldsymbol{\beta}} f\left(\mathbf{C}_{ij}, \tilde{\boldsymbol{\beta}}_{i\ell(j)}\right) \right| \left| \widehat{\boldsymbol{\beta}}_{i\ell(j)} - \boldsymbol{\beta}_i^* \right|
\end{aligned} \tag{32}$$

The first term of inequality (32) goes to zero as $r \to \infty$. The second term goes to zero by the SLLN. By assumption $\sup_{\boldsymbol{\beta}_i \in \mathscr{U}} \left| \partial f\left(\mathbf{c}, \boldsymbol{\beta}_i\right) / \partial \left(\boldsymbol{\beta}_i\right)_d \right| \leq W(\mathbf{c})$, then third term has

$$\frac{1}{r} \sum_{j=r_1+1}^{r} \left| \nabla_{\boldsymbol{\beta}} f\left(\mathbf{C}_{ij}, \tilde{\boldsymbol{\beta}}_{i\ell(j)}\right) \right| \left| \widehat{\boldsymbol{\beta}}_{i\ell(j)} - \boldsymbol{\beta}_i^* \right| < \frac{p}{r} \sum_{j=r_1+1}^{r} W(\mathbf{C}_{ij}) \left| \widehat{\boldsymbol{\beta}}_{i\ell(j)} - \boldsymbol{\beta}_i^* \right|.$$

By Assumption 3, $\mathrm{E}\left[(W(\mathbf{C}_{ij}))^2\right] < \infty$, then by SLLN, $\frac{1}{r} \sum_{j=1}^{r} W(\mathbf{C}_{ij}) \xrightarrow{\text{a.s.}} \mathrm{E}[W(\mathbf{C}_{ij})] \triangleq w_i > 0$. That is, there exists another set $E_2 \in \mathcal{F}$ with $\Pr\{E_2\} = 0$ such that $\forall \omega \in E_2^c$ and $\forall \epsilon > 0$, $\exists r_2 \in \mathbb{Z}^+$ such that $\forall r > r_2$,

$$\left| \frac{1}{r} \sum_{j=1}^{r} W(\mathbf{C}_{ij}) - w_i \right| < \epsilon.$$

Combine above inequality with (31),

$$\begin{aligned}\epsilon^2 &> \left|\frac{1}{r}\sum_{j=1}^{r} W(\mathbf{C}_{ij}) - w_i\right| \left|\widehat{\boldsymbol{\beta}}_{i\ell(j)} - \boldsymbol{\beta}_i^*\right| \\ &\geq \frac{1}{r}\sum_{j=1}^{r} W(\mathbf{C}_{ij}) \left|\widehat{\boldsymbol{\beta}}_{i\ell(j)} - \boldsymbol{\beta}_i^*\right| - w_i \left|\widehat{\boldsymbol{\beta}}_{i\ell(j)} - \boldsymbol{\beta}_i^*\right| \\ &> \frac{1}{r}\sum_{j=1}^{r} W(\mathbf{C}_{ij}) \left|\widehat{\boldsymbol{\beta}}_{i\ell(j)} - \boldsymbol{\beta}_i^*\right| - w_i\epsilon,\end{aligned}$$

which implies that

$$\frac{1}{r}\sum_{j=r_1+1}^{r} W(\mathbf{C}_{ij}) \left|\widehat{\boldsymbol{\beta}}_{i\ell(j)} - \boldsymbol{\beta}_i^*\right| \xrightarrow{\text{a.s.}} 0.$$

So

$$\frac{1}{r}\sum_{j=r_1+1}^{r} \left|\nabla_{\boldsymbol{\beta}} f\left(\mathbf{C}_{ij}, \tilde{\boldsymbol{\beta}}_{i\ell(j)}\right)\right| \left|\widehat{\boldsymbol{\beta}}_{i\ell(j)} - \boldsymbol{\beta}_i^*\right| \xrightarrow{\text{a.s.}} 0.$$

Then,

$$\left|\frac{1}{r}\sum_{j=1}^{r} f\left(\mathbf{C}_{ij}, \widehat{\boldsymbol{\beta}}_{i\ell(j)}\right)\right| \xrightarrow{\text{a.s.}} 0.$$

□

To prove the CLT of the nonlinear CV estimator $\bar{\theta}_i(r)$, we need the results in Bradley (1992), in which we first define some new notation. Let $\mathscr{A}$ and $\mathscr{B}$ be two σ-field in our probability space, and define the maximal correlation,

$$\rho(\mathscr{A}, \mathscr{B}) := \sup_{V,W}\left\{\frac{|\mathrm{E}[VW] - \mathrm{E}[V]\mathrm{E}[W]|}{\sqrt{\mathrm{Var}[V]\mathrm{Var}[W]}}\right\},$$

where the supremum is taken over all square-integrable random variables V and W which are $\mathscr{A}$-measurable and $\mathscr{B}$-measurable, respectively.

Define $\widehat{\theta} := \{\widehat{\theta}_j, j \in \mathbb{Z}^+\}$, which is assumed to be a centered and strictly stationary random variable sequence. Let $S \subset \mathbb{Z}^+$ and $D \subset \mathbb{Z}^+$ be any two nonempty disjoint sets, and we use the abbreviation

$$\rho(S, D) := \rho\Big(\sigma\Big(\{\widehat{\theta}_\iota, \iota \in S\}\Big), \sigma\left(\{\widehat{\theta}_\tau, \tau \in D\}\right)\Big),$$

where $\sigma(\cdot)$ denotes the σ-field generated by $(\cdot)$. For each real number $r \geq 1$, define

$$\rho^*(r) := \sup_{S,D} \rho(S, D),$$

where the supremum is taken over all pairs of nonempty disjoint subsets $S, D \subset \mathbb{Z}^+$ such that $\text{dist}(S, D) \geq r$ with $\text{dist}(S, D) := \min_{\iota \in S, \tau \in D} |\tau - \iota|$.

Lemma 7 (Theorem 4 in Bradley (1992))**.** *Suppose that* $\widehat{\theta} := \{\widehat{\theta}_j, j \in \mathbb{Z}^+\}$ *is a centered and strictly stationary random variable sequence such that* $0 < \mathrm{E}[\widehat{\theta}_1^2] < \infty$, $\rho^*(r) \to 0$ *as* $r \to \infty$, *and the continuous spectral density* $g(\cdot)$ *of* $\widehat{\theta}$ *satisfies* $g(1) > 0$. *Then, as* $r \to \infty$, *we have* $\sigma_r^2 := \mathrm{Var}\left[\widehat{\theta}_1 + \widehat{\theta}_2 + \cdots + \widehat{\theta}_r\right] \to \infty$ *and that*

$$\frac{\sum_{j=1}^{r} \widehat{\theta}_j}{\sigma_r} \Rightarrow N(0, 1).$$

In order to apply the result in Lemma 7, we need to establish another important result. Define $\mathrm{Corr}(V, W)$ to be the correlation between two random variables V and W, and we would like to show that the controlled observations $\{\widehat{\theta}_{ij}, j \in \mathbb{Z}^+\}$ satisfy that $\mathrm{Corr}(\widehat{\theta}_{ij}, \widehat{\theta}_{i,j+r}) \to 0$ as $r \to \infty$. That is, $\widehat{\theta}_{ij}$ and $\widehat{\theta}_{i,j+r}$ become essentially independent as r becomes large.

Lemma 8. *Under Assumptions 1–4, if for any fixed* i *and all* j, $0 < \mathrm{Var}[\widehat{\theta}_{ij}] < \infty$. *Then, for the controlled observations* $\{\widehat{\theta}_{ij}, j \in \mathbb{Z}^+\}$, *we know that* $\mathrm{Corr}(\widehat{\theta}_{ij}, \widehat{\theta}_{i,j+r}) \to 0$ *as* $r \to \infty$.

Proof. Without loss of generality, we assume that $\widehat{\theta}_{ij}$ and $\widehat{\theta}_{i,j+r}$ belong to different stages (saying, ℓ_1 and ℓ_2, where $\ell_1 < \ell_2$). Then, we can write the covariance of $\widehat{\theta}_{ij}$ and $\widehat{\theta}_{i,j+r}$ as

$$\begin{aligned}
\mathrm{Cov}\big[\widehat{\theta}_{ij}, \widehat{\theta}_{i,j+r}\big] &= \mathrm{Cov}\left[Y_{ij} - f\big(\mathbf{C}_{ij}, \widehat{\boldsymbol{\beta}}_{i\ell_1}\big), Y_{i,j+r} - f\big(\mathbf{C}_{i,j+r}, \widehat{\boldsymbol{\beta}}_{i\ell_2}\big)\right] \\
&= \mathrm{Cov}\left[f\big(\mathbf{C}_{ij}, \widehat{\boldsymbol{\beta}}_{i\ell_1}\big), f\big(\mathbf{C}_{i,j+r}, \widehat{\boldsymbol{\beta}}_{i\ell_2}\big)\right] - \mathrm{Cov}\left[Y_{ij}, f\big(\mathbf{C}_{i,j+r}, \widehat{\boldsymbol{\beta}}_{i\ell_2}\big)\right] \qquad (33) \\
&= \mathrm{E}\left[f\big(\mathbf{C}_{ij}, \widehat{\boldsymbol{\beta}}_{i\ell_1}\big) f\big(\mathbf{C}_{i,j+r}, \widehat{\boldsymbol{\beta}}_{i\ell_2}\big)\right] - \mathrm{E}\left[Y_{ij} f\big(\mathbf{C}_{i,j+r}, \widehat{\boldsymbol{\beta}}_{i\ell_2}\big)\right], \qquad (34)
\end{aligned}$$

where $\ell_1 = \max\left\{0, \lceil \frac{j-n_0}{L} \rceil\right\}$, and $\ell_2 = \max\left\{0, \lceil \frac{j+r-n_0}{L} \rceil\right\}$.

We can see that Equation (33) holds since $\mathrm{Cov}\big[Y_{ij}, Y_{i,j+r}\big] = \mathrm{Cov}\big[f\big(\mathbf{C}_{ij}, \widehat{\boldsymbol{\beta}}_{i\ell_1}\big), Y_{i,j+r}\big] = 0$. The other covariance terms remain due to the correlation introduced by overlapping samples. Equation (34) holds because $\mathrm{E}[f(\cdot, \cdot)] = 0$ under Assumption 1. In addition, there are countless possible forms of the nonlinear function $f(\mathbf{C}_{ij}, \widehat{\boldsymbol{\beta}}_i)$, therefore we use Taylor approximation to proceed the derivation. Under appropriate differentiability assumptions (i.e., Assumptions 2 and 3), the first-order Taylor expansion of $f\left(\mathbf{C}_{ij}, \widehat{\boldsymbol{\beta}}_i\right)$ centered at $\widehat{\boldsymbol{\beta}}_i = \boldsymbol{\beta}_i^*$ is as follows,

$$f\left(\mathbf{C}_{ij}, \widehat{\boldsymbol{\beta}}_i\right) = f\left(\mathbf{C}_{ij}, \boldsymbol{\beta}_i^*\right) + \nabla_{\boldsymbol{\beta}} f\left(\mathbf{C}_{ij}, \widetilde{\boldsymbol{\beta}}_i^*\right)^{\mathsf{T}} \left(\widehat{\boldsymbol{\beta}}_i - \boldsymbol{\beta}_i^*\right)$$

where $\widetilde{\boldsymbol{\beta}}_i^*$ is located between $\widehat{\boldsymbol{\beta}}_i$ and $\boldsymbol{\beta}_i^*$ and the operator $\nabla_{\boldsymbol{\beta}} f\left(\mathbf{C}_{ij}, \boldsymbol{\beta}_i\right)$ is taken derivative with respect to the second component $\boldsymbol{\beta}_i$ in the function $f(\cdot, \cdot)$.

Then, based on Equation (34) we can rewrite

$$\begin{aligned}
\mathrm{Cov}\big[\widehat{\theta}_{ij}, \widehat{\theta}_{i,j+r}\big] &= \mathrm{E}\Bigg[\Bigg[f\left(\mathbf{C}_{ij}, \boldsymbol{\beta}_i^*\right) + \nabla_{\boldsymbol{\beta}} f\left(\mathbf{C}_{ij}, \widetilde{\boldsymbol{\beta}}_i^*\right)^{\mathsf{T}} (\widehat{\boldsymbol{\beta}}_{i\ell_1} - \boldsymbol{\beta}_i^*)\Bigg] \\
&\qquad \times \left[f\left(\mathbf{C}_{i,j+r}, \boldsymbol{\beta}_i^*\right) + \nabla_{\boldsymbol{\beta}} f\left(\mathbf{C}_{i,j+r}, \breve{\boldsymbol{\beta}}_i^*\right)^{\mathsf{T}} (\widehat{\boldsymbol{\beta}}_{i\ell_2} - \boldsymbol{\beta}_i^*)\right]\Bigg] \\
&\quad -\mathrm{E}\left[Y_{ij}\left[f\left(\mathbf{C}_{i,j+r}, \boldsymbol{\beta}_i^*\right) + \nabla_{\boldsymbol{\beta}} f\left(\mathbf{C}_{i,j+r}, \breve{\boldsymbol{\beta}}_i^*\right)^{\mathsf{T}} (\widehat{\boldsymbol{\beta}}_{i\ell_2} - \boldsymbol{\beta}_i^*)\right]\right] \qquad (35) \\
&= \mathrm{E}\Big[\nabla_{\boldsymbol{\beta}} f\left(\mathbf{C}_{ij}, \widetilde{\boldsymbol{\beta}}_i^*\right)^{\mathsf{T}} (\widehat{\boldsymbol{\beta}}_{i\ell_1} - \boldsymbol{\beta}_i^*) \nabla_{\boldsymbol{\beta}} f\left(\mathbf{C}_{i,j+r}, \breve{\boldsymbol{\beta}}_i^*\right)^{\mathsf{T}} (\widehat{\boldsymbol{\beta}}_{i\ell_2} - \boldsymbol{\beta}_i^*)\Big] \\
&\quad +\mathrm{E}\left[f\left(\mathbf{C}_{ij}, \boldsymbol{\beta}_i^*\right) \nabla_{\boldsymbol{\beta}} f\left(\mathbf{C}_{i,j+r}, \breve{\boldsymbol{\beta}}_i^*\right)^{\mathsf{T}} (\widehat{\boldsymbol{\beta}}_{i\ell_2} - \boldsymbol{\beta}_i^*)\right] \\
&\quad -\mathrm{E}\left[Y_{ij} \nabla_{\boldsymbol{\beta}} f\left(\mathbf{C}_{i,j+r}, \breve{\boldsymbol{\beta}}_i^*\right)^{\mathsf{T}} (\widehat{\boldsymbol{\beta}}_{i\ell_2} - \boldsymbol{\beta}_i^*)\right]. \qquad (36)
\end{aligned}$$

Equation (35) follows from the application of Taylor approximation, where $\widetilde{\boldsymbol{\beta}}_i^*$ is located between $\widehat{\boldsymbol{\beta}}_{i\ell_1}$ and $\boldsymbol{\beta}_i^*$, and $\breve{\boldsymbol{\beta}}_i^*$ is located between $\widehat{\boldsymbol{\beta}}_{i\ell_2}$ and $\boldsymbol{\beta}_i^*$. Since that $\mathbf{C}_{i,j+r}$ is independent of $\widehat{\boldsymbol{\beta}}_{i\ell_1}$, $\mathbf{C}_{i,j+r}$ is independent of $\widetilde{\boldsymbol{\beta}}_i^*$, and Y_{ij} is independent of $\mathbf{C}_{i,j+r}$, we simplify Equation (35) to Equation (36) by Assumption 1 to zero out the expectation of the nonlinear function.

Next, consider the first term on the RHS of Equation (36). By Assumption 2, $\mathbf{B} \subset \mathbb{R}^p$ is compact, so $\|\boldsymbol{\beta}_i - \boldsymbol{\beta}_i'\| \leq K$, $\forall \boldsymbol{\beta}_i, \boldsymbol{\beta}_i' \in \mathbf{B}$, for a large enough constant $K < \infty$. On the other hand, by Assumption 3, $\sup_{\boldsymbol{\beta}_i \in \mathscr{U}} |\partial f(\mathbf{c}, \boldsymbol{\beta}_i)/\partial \left(\boldsymbol{\beta}_i\right)_d| \leq W(\mathbf{c})$, then we have

$$\left|\nabla_{\boldsymbol{\beta}} f\left(\mathbf{C}_{ij}, \widetilde{\boldsymbol{\beta}}_i^*\right)^{\mathsf{T}} (\widehat{\boldsymbol{\beta}}_{i\ell_1} - \boldsymbol{\beta}_i^*)\right| \leq pKW(\mathbf{C}_{ij})$$

and

$$\left|\nabla_{\boldsymbol{\beta}} f\left(\mathbf{C}_{ij}, \breve{\boldsymbol{\beta}}_i^*\right)^{\mathsf{T}} (\widehat{\boldsymbol{\beta}}_{i\ell_2} - \boldsymbol{\beta}_i^*)\right| \leq pKW(\mathbf{C}_{ij}).$$

By the condition $\mathrm{E}\big[\left(W(\mathbf{C}_{ij})\right)^2\big] < \infty$ and the dominated convergence theorem, then the limit and the expectation can be interchanged. That is, as $r \to \infty$ we obtain that

$$\begin{aligned}
&\lim_{r\to\infty} \mathrm{E}\Big[\nabla_{\boldsymbol{\beta}} f\left(\mathbf{C}_{ij}, \widetilde{\boldsymbol{\beta}}_i^*\right)^{\mathsf{T}} (\widehat{\boldsymbol{\beta}}_{i\ell_1} - \boldsymbol{\beta}_i^*) \nabla_{\boldsymbol{\beta}} f\left(\mathbf{C}_{i,j+r}, \breve{\boldsymbol{\beta}}_i^*\right)^{\mathsf{T}} (\widehat{\boldsymbol{\beta}}_{i\ell_2} - \boldsymbol{\beta}_i^*)\Big] \\
= \ & \mathrm{E}\Big[\lim_{r\to\infty} \nabla_{\boldsymbol{\beta}} f\left(\mathbf{C}_{ij}, \widetilde{\boldsymbol{\beta}}_i^*\right)^{\mathsf{T}} (\widehat{\boldsymbol{\beta}}_{i\ell_1} - \boldsymbol{\beta}_i^*) \nabla_{\boldsymbol{\beta}} f\left(\mathbf{C}_{i,j+r}, \breve{\boldsymbol{\beta}}_i^*\right)^{\mathsf{T}} (\widehat{\boldsymbol{\beta}}_{i\ell_2} - \boldsymbol{\beta}_i^*)\Big] \\
= \ & 0.
\end{aligned}$$

The last equality holds because $\ell_2 \to \infty$ and $\widehat{\boldsymbol{\beta}}_{i\ell_2} \to \boldsymbol{\beta}_i^*$ as $r \to \infty$.

We now consider the second term on the RHS of Equation (36). Notice that

$$\left| f\left(\mathbf{C}_{ij}, \boldsymbol{\beta}_i^*\right) \nabla_{\boldsymbol{\beta}} f\left(\mathbf{C}_{i,j+r}, \breve{\boldsymbol{\beta}}_i^*\right)^{\intercal} \left(\widehat{\boldsymbol{\beta}}_{i\ell_2} - \boldsymbol{\beta}_i^*\right) \right| \leq pK \left| f\left(\mathbf{C}_{ij}, \boldsymbol{\beta}_i^*\right) \right| W(\mathbf{C}_{i,j+r}).$$

By Cauchy-Schwarz Inequality and Assumption 1,

$$\left(\mathrm{E}\left[\left| f\left(\mathbf{C}_{ij}, \boldsymbol{\beta}_i^*\right) \right| W(\mathbf{C}_{i,j+r})\right]\right)^2 \leq \mathrm{E}\left[\left(f\left(\mathbf{C}_{ij}, \boldsymbol{\beta}_i^*\right)\right)^2\right] \mathrm{E}\left[\left(W(\mathbf{C}_{i,j+r})\right)^2\right] < \infty.$$

Then, by the dominated convergence theorem again, we know that

$$\begin{aligned} & \lim_{r\to\infty} \mathrm{E}\left[f\left(\mathbf{C}_{ij}, \boldsymbol{\beta}_i^*\right) \nabla_{\boldsymbol{\beta}} f\left(\mathbf{C}_{i,j+r}, \breve{\boldsymbol{\beta}}_i^*\right)^{\intercal} \left(\widehat{\boldsymbol{\beta}}_{i\ell_2} - \boldsymbol{\beta}_i^*\right) \right] \\ = & \mathrm{E}\left[\lim_{r\to\infty} f\left(\mathbf{C}_{ij}, \boldsymbol{\beta}_i^*\right) \nabla_{\boldsymbol{\beta}} f\left(\mathbf{C}_{i,j+r}, \breve{\boldsymbol{\beta}}_i^*\right)^{\intercal} \left(\widehat{\boldsymbol{\beta}}_{i\ell_2} - \boldsymbol{\beta}_i^*\right) \right] \\ = & 0. \end{aligned}$$

Similarly, we consider the third term on the RHS of Equation (36). Notice that

$$\begin{aligned} \left| Y_{ij} \nabla_{\boldsymbol{\beta}} f\left(\mathbf{C}_{i,j+r}, \breve{\boldsymbol{\beta}}_i^*\right)^{\intercal} \left(\widehat{\boldsymbol{\beta}}_{i\ell_2} - \boldsymbol{\beta}_i^*\right) \right| & \leq |Y_{ij}| \left| \nabla_{\boldsymbol{\beta}} f\left(\mathbf{C}_{i,j+r}, \breve{\boldsymbol{\beta}}_i^*\right)^{\intercal} \left(\widehat{\boldsymbol{\beta}}_{i\ell_2} - \boldsymbol{\beta}_i^*\right) \right| \\ & \leq pK |Y_{ij}| W(\mathbf{C}_{i,j+r}). \end{aligned}$$

By Cauchy-Schwarz Inequality and Assumption 1 again,

$$\left(\mathrm{E}\left[|Y_{ij}| W(\mathbf{C}_{i,j+r})\right]\right)^2 \leq \mathrm{E}\left[Y_{ij}^2\right] \mathrm{E}\left[\left(W(\mathbf{C}_{i,j+r})\right)^2\right] < \infty.$$

Then, by the dominated convergence theorem again,

$$\begin{aligned} & \lim_{r\to\infty} \mathrm{E}\left[Y_{ij} \nabla_{\boldsymbol{\beta}} f\left(\mathbf{C}_{i,j+r}, \breve{\boldsymbol{\beta}}_i^*\right)^{\intercal} \left(\widehat{\boldsymbol{\beta}}_{i\ell_2} - \boldsymbol{\beta}_i^*\right) \right] \\ = & \mathrm{E}\left[\lim_{r\to\infty} Y_{ij} \nabla_{\boldsymbol{\beta}} f\left(\mathbf{C}_{i,j+r}, \breve{\boldsymbol{\beta}}_i^*\right)^{\intercal} \left(\widehat{\boldsymbol{\beta}}_{i\ell_2} - \boldsymbol{\beta}_i^*\right) \right] \\ = & 0. \end{aligned}$$

Finally, we can conclude that $\mathrm{Cov}[\widehat{\theta}_{ij}, \widehat{\theta}_{i,j+r}] \to 0$ as $r \to \infty$. Since $0 < \mathrm{Var}[\widehat{\theta}_{ij}] < \infty$ for any fixed i and all j, then we have $\mathrm{Corr}(\widehat{\theta}_{ij}, \widehat{\theta}_{i,j+r}) \to 0$ as $r \to \infty$.

□

Now, we are ready to prove the CLT result in Proposition 4 based on Lemmas 7 and 8.

Proof of Proposition 4 (ii). Define $\widetilde{\theta}_{ij} := \widehat{\theta}_{ij} - \theta_i$ and $\widetilde{\theta}_i := \left\{\widetilde{\theta}_{ij}, j \in \mathbb{Z}^+\right\}$. As $\left\{\widehat{\theta}_{ij}, j \in \mathbb{Z}^+\right\}$ is a strictly stationary random variable sequence, $\widetilde{\theta}_i$ is a centered and strictly stationary random

variable sequence. According to Lemma 7, it suffices to verify the conditions $\rho^*(r) \to 0$ as $r \to \infty$ and $g(1) > 0$.

We first consider $\rho^*(r)$. For ease of notation, we drop off the subscript i, and consider each alternative independently. According to Lemma 8, we know that $\forall j$, $\mathrm{Corr}(\tilde{\theta}_j, \tilde{\theta}_{j+r}) \to 0$ as $r \to \infty$. So $\forall \epsilon > 0$, there exists a r_0 such that $\mathrm{Corr}(\tilde{\theta}_j, \tilde{\theta}_{j+r}) < \epsilon$ for $r > r_0$.

By the definition, $\rho^*(r) = \sup \rho(S, D)$, where $\rho(S, D) = \rho(\sigma(\tilde{\theta}_k, k \in S), \sigma(\tilde{\theta}_k, k \in D))$, and $\mathrm{dist}(S, D) = \min_{\iota \in S, \tau \in D} |\iota - \tau|$. Now for $\forall S, D \subset \mathbb{Z}^+$, if $\mathrm{dist}(S, D) = r > r_0$, then $\forall \iota \in S$ and $\forall \tau \in D$, $|\iota - \tau| > r_0$. Thus, $\mathrm{Corr}(\tilde{\theta}_\iota, \tilde{\theta}_\tau) < \epsilon$. That is, when $\mathrm{dist}(S, D) = r > r_0$, $\rho(S, D) < \epsilon$. Then $\rho^*(r) = \sup \rho(S, D) < \epsilon$. Since $\epsilon > 0$ is arbitrary, $\rho^*(r) \to 0$ as $r \to \infty$.

Next, we consider the spectral density $g(1)$. According to the definition of the spectral density,

$$g(e^{\imath\lambda}) = \lim_{r\to\infty} g_r(e^{\imath\lambda}) = \lim_{r\to\infty} \frac{1}{r} \mathrm{E}\left[\left|\sum_{k=1}^{r} e^{-\imath k\lambda} \tilde{\theta}_{ik}\right|^2\right].$$

Then let $\lambda = 0$, we have

$$g(1) = \lim_{r\to\infty} \frac{1}{r} \mathrm{E}\left[\left|\sum_{k=1}^{r} \tilde{\theta}_{ik}\right|^2\right].$$

By the condition that $\mathrm{E}[(\sum_{i=1}^{r}(\hat{\theta}_{ij} - \theta_i))^2] = O(r)$, i.e., $\mathrm{E}[(\sum_{k=1}^{r} \tilde{\theta}_{ik})^2] = O(r)$, and it implies that $g(1) \neq 0$, i.e., $g(1) > 0$.

With all the conditions in Lemma 7 being verified and adding back the subscript i, we have

$$\sum_{j=1}^{r} \widetilde{\theta}_{ij} / \sigma_{ir} \Rightarrow N(0, 1),$$

where $\sigma_{ir}^2 = r^2 \mathrm{Var}\left[\bar{\theta}_i(r)\right]$ and $\sum_{j=1}^{r} \widetilde{\theta}_{ij} = r(\bar{\theta}_i(r) - \theta_i)$. Therefore, Proposition 4 (ii) has been proved. □

C Two Existing Fully Sequential Selection Procedures

In this section, we present the classical fully sequential selection procedure, i.e., $\mathcal{KN}$ procedure in Kim and Nelson (2001), as well as the fully sequential selection procedure with linear CV, i.e., $\mathcal{TN}$ procedure in Tsai and Nelson (2010).

We assume that CRN is not used in both procedures, and that the linear CV model holds in $\mathcal{TN}$. In addition, an independent preliminary stage is used in $\mathcal{TN}$ to estimate $\widehat{\boldsymbol{\beta}}_i$, and then its value is fixed in the subsequent elimination process. In this way, all the controlled observations for constructing the CS estimator are i.i.d. as conditioned on $\widehat{\boldsymbol{\beta}}_i$. By doing so, we can guarantee

a finite-time statistical validity for $\mathcal{TN}$. For both procedures, we can show that, under the IZ formulation, $\Pr\{\text{select system } 1 | \theta_1 \geq \theta_2 + \delta\} \geq 1 - \alpha$. Below are the detailed descriptions of two procedures.

Procedure 2 ($\mathcal{KN}$ Procedure in Kim and Nelson (2001))**.**

Step 0. Setup: Select confidence level $1/k < 1 - \alpha < 1$, IZ parameter $\delta > 0$, and the first-stage sample size $n_0 \geq 2$. Let $g^2 = 2\eta \times (n_0 - 1)$, where

$$\eta = \frac{1}{2}\left\{\left[2\left(1 - (1-\alpha)^{\frac{1}{k-1}}\right)\right]^{-2/(n_0-1)} - 1\right\}.$$

Step 1. Initialization: Let $I = \{1, 2, \ldots, k\}$ be the set of systems still in contention. For each system $i \in I$, perform independent sampling to generate $\{Y_{ij}, j = 1, 2, \ldots, n_0\}$ and compute the sample variance

$$S_i^2 = \frac{1}{n_0 - 1}\sum_{j=1}^{n_0}(Y_{ij} - \bar{Y}_i(n_0))^2.$$

Let r be the observation counter. Set $r = n_0$.

Step 2. Elimination: Set $I^{\text{old}} = I$. Let

$$I = I^{\text{old}} \setminus \left\{i \in I^{\text{old}} : \bar{Y}_i(r) - \bar{Y}_h(r) < \min\left\{0, -W_{ih}(r) + \frac{\delta}{2}\right\} \text{ for some } h \in I^{\text{old}} \text{ and } h \neq i\right\},$$

where

$$W_{ih}(r) = \frac{g^2 \cdot [S_i^2 + S_h^2]}{2\delta r}$$

and $A \setminus B = \{x : x \in A \text{ and } x \notin B\}$.

Step 3. Stopping Rule: If $|I| = 1$, then stop and select the system whose index is in I as the best. Otherwise, take one additional output $Y_{i,r+1}$ from each system $i \in I$, set $r = r + 1$ and go to **Step 2**.

Procedure 3 ($\mathcal{TN}$ Procedure in Tsai and Nelson (2010))**.**

Step 0. Setup: Select confidence level $1/k < 1 - \alpha < 1$, IZ parameter $\delta > 0$, preliminary-stage sample size $m_0 > \max_{i=1,\ldots,k} q_i + 2$, and first-stage sample size $n_0 \geq 2$. Let $g^2 = 2\eta \times (n_0 - 1)$, where

$$\eta = \frac{1}{2}\left\{\left[2\left(1 - (1-\alpha)^{\frac{1}{k-1}}\right)\right]^{-2/(n_0-1)} - 1\right\}.$$

Step 1. Initialization: Let $I = \{1, 2, \ldots, k\}$ be the set of systems still in contention. For each system $i \in I$, generate $\{(Y_{ij}, \mathbf{C}_{ij}), j = 1, 2, \ldots, m_0\}$ and then compute the estimator $\widehat{\boldsymbol{\beta}}_i(m_0)$ according to Equation (2). For each system $i \in I$, perform additional independent sampling to generate $\{(Y_{ij}, \mathbf{C}_{ij}), j = m_0+1, m_0+2, \ldots, m_0+n_0\}$ and compute $S_i^2[m_0, n_0]$ (see Section 2.1). Let r be the observation counter. Set $r = n_0$.

Step 2. Elimination: Calculate the sample mean of the first r observations (which starts after the preliminary stage) from system i

$$\widehat{\theta}_i(r) = \bar{Y}_i(r) - \left(\bar{\mathbf{C}}_i(r) - \boldsymbol{\mu}_i\right)^{\mathrm{T}} \widehat{\boldsymbol{\beta}}_i(m_0).$$

Set $I^{\text{old}} = I$. Let

$$I = I^{\text{old}} \setminus \left\{ i \in I^{\text{old}} : \widehat{\theta}_i(r) - \widehat{\theta}_h(r) < \min\left\{0, -W_{ih}(r) + \frac{\delta}{2}\right\} \text{ for some } h \in I^{\text{old}} \text{ and } h \neq i \right\},$$

where

$$W_{ih}(r) = \frac{g^2 \cdot [S_i^2[m_0, n_0] + S_h^2[m_0, n_0]]}{2\delta r}$$

and $A \setminus B = \{x : x \in A \text{ and } x \notin B\}$.

Step 3. Stopping Rule: If $|I| = 1$, then stop and select the system whose index is in I as the best. Otherwise, let $r = r + 1$, take one additional observation from system $i \in I$, and go to **Step 2**.